Arthur T. Benjamin is Professor of Mathematics at Harvey Mudd College, where he has taught since 1989. He earned a B.Sc. from Carnegie Mellon University in 1983 and a Ph.D. in Mathematical Sciences from Johns Hopkins University in 1989. The Mathematical Association of America honored him with regional and national awards for distinguished teaching in 1999 and 2000 and named him the George Pólya Lecturer for 2006–2008.

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The Joy of Mathematics

Taught by: Professor Arthur T. Benjamin
Harvey Mudd College

Part 1

Course Guidebook

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Arthur T. Benjamin, Ph.D.

Professor of Mathematics, Harvey Mudd College

Arthur T. Benjamin is a Professor of Mathematics at Harvey Mudd College. He graduated from Carnegie Mellon University in 1983, where he earned a B.S. in Applied Mathematics with university honors. He received his Ph.D. in Mathematical Sciences in 1989 from Johns Hopkins University, where he was supported by a National Science Foundation graduate fellowship and a Rufus P. Isaacs fellowship. Since 1989, Dr. Benjamin has been a faculty member of the Mathematics Department at Harvey Mudd College, where he has served as department chair. He has spent sabbatical visits at Caltech, Brandeis University, and University of New South Wales in Sydney, Australia.

In 1999, Professor Benjamin received the Southern California Section of the Mathematical Association of America (MAA) Award for Distinguished College or University Teaching of Mathematics, and in 2000, he received the MAA Deborah and Franklin Tepper Haimo National Award for Distinguished College or University Teaching of Mathematics. He was named the 2006–2008 George Pólya Lecturer by the MAA.

Dr. Benjamin's research interests include combinatorics, game theory, and number theory, with a special fondness for Fibonacci numbers. Many of these ideas appear in his book (co-authored with Jennifer Quinn), *Proofs That Really Count: The Art of Combinatorial Proof* published by the MAA. In 2006, that book received the Beckenbach Book Prize by the MAA. Professors Benjamin and Quinn are the co-editors of *Math Horizons* magazine, published by MAA and enjoyed by more than 20,000 readers, mostly undergraduate math students and their teachers.

Professor Benjamin is also a professional magician. He has given more than 1,000 "mathemagics" shows to audiences all over the world (from primary schools to scientific conferences), where he demonstrates and explains his calculating talents. His techniques are explained in his book *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks*. Prolific math and science writer Martin Gardner calls it "the clearest, simplest, most entertaining, and best book yet on the art of calculating in your head." An avid games player, Dr. Benjamin was winner of the American Backgammon Tour in 1997.

Professor Benjamin has appeared on dozens of television and radio programs, including the *Today Show*, CNN, and National Public Radio. He has been featured in *Scientific American*, *Omni*, *Discover*, *People*, *Esquire*, *The New York Times*, *The Los Angeles Times*, and *Reader's Digest*. In 2005, *Reader's Digest* called him "America's Best Math Whiz."

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The Joy of Mathematics

Scope:

For most people, mathematics is little more than counting: basic arithmetic and bookkeeping. People might recognize that numbers are important, but most cannot fathom how anyone could find mathematics to be a subject that can be described by such adjectives as *joyful*, *beautiful*, *creative*, *inspiring*, or *fun*. This course aims to show how mathematics—from the simplest notions of numbers and counting to the more complex ideas of calculus, imaginary numbers, and infinity—is indeed a great source of joy.

Throughout most of our education, mathematics is used as an exercise in disciplined thinking. If you follow certain procedures carefully, you will arrive at the right answer. Although this approach has its value, I think that not enough attention is given to teaching math as an opportunity to explore creative thinking. Indeed, it's marvelous to see how often we can take a problem, even a simple arithmetic problem, solve it lots of different ways, and always arrive at the same answer. This internal consistency of mathematics is beautiful. When numbers are organized in other ways, such as in Pascal's triangle or the Fibonacci sequence, then even more beautiful patterns emerge, most of which can be appreciated from many different perspectives. Learning that there is more than one way to solve a problem or understand a pattern is a valuable life lesson in itself.

Another special quality of mathematics, one that separates it from other academic disciplines, is its ability to achieve absolute certainty. Once the definitions and rules of the game (the rules of logic) are established, you can reach indisputable conclusions. For example, mathematics can prove, beyond a shadow of a doubt, that there are infinitely many prime numbers and that the Pythagorean theorem (concerning the lengths of the sides of a right triangle) is absolutely true, now and forever. It can also "prove the impossible," from easy statements, such as "The sum of two even numbers is never an odd number," to harder ones, such as "The digits of pi (π) will never repeat." Scientific theories are constantly being refined and improved and, occasionally, tossed aside in light of better evidence. But a mathematical theorem is true forever. We still marvel over the brilliant logical arguments put forward by the ancient Greek mathematicians more than 2,000 years ago.

From backgammon and bridge to chess and poker, many popular games utilize math in some way. By understanding math, especially probability and combinatorics (the mathematics of counting), you can become a better game player and win more.

Of course, there is more to love about math besides using it to win games, or solve problems, or prove something to be true. Within the universe of numbers,

there are intriguing patterns and mysteries waiting to be explored. This course will reveal some of these patterns to you.

In choosing material for this course, I wanted to make sure to cover the highlights of the traditional high school mathematics curriculum of algebra, geometry, trigonometry, and calculus, but in a nontraditional way. I will introduce you to some of the great numbers of mathematics, including π, e, i, 9, the numbers in Pascal's triangle, and (my personal favorites) the Fibonacci numbers. Toward the end of the course, as we explore notions of infinity, infinite series, and calculus, the material becomes a little more challenging, but the rewards and surprises are even greater.

Although we will get our hands dirty playing with numbers, manipulating algebraic expressions, and exploring many of the fundamental theorems in mathematics (including the fundamental theorems of arithmetic, algebra, and calculus), we will also have fun along the way, not only with the occasional song, dance, poem, and lots of bad jokes, but also with three lectures exploring applications to games and gambling. Aside from being a professor of mathematics, I have more than 30 years experience as a professional magician, and I try to infuse a little bit of magic in everything I teach. In fact, the last lesson of the course (which you could watch first, if you want) is on the joy of mathematical magic.

Mathematics is food for the brain. It helps you think precisely, decisively, and creatively and helps you look at the world from multiple perspectives. Naturally it comes in handy when dealing with numbers directly, such as when you're shopping around for the best bargain or trying to understand the statistics you read in the newspaper. But I hope that you also come away from this course with a new way to experience beauty, in the form of a surprising pattern or an elegant logical argument. Many people find joy in fine music, poetry, and other works of art, and mathematics offers joys that I hope you, too, will learn to experience. If Elizabeth Barrett Browning had been a mathematician, she might have said, "How do I count thee? Let me love the ways!"

Lecture One

The Joy of Math—The Big Picture

Scope: For many people, the phrase "joy of mathematics" sounds like a contradiction in terms. In school, many people experience mathematics as a subject filled with confusing rules and mindless procedures that do not seem to have much of a purpose. But I love mathematics, and my goal in these lectures is for my passion to be contagious, so that you will feel the same way. Mathematics is the science of patterns, and some people love mathematics simply for the beautiful patterns that can be discovered when playing with numbers, geometry, and algebra. Others enjoy mathematics because of its utility. Mathematics can be used to help us make decisions, model natural phenomena, and sometimes, make predictions in a random, unpredictable world. I love mathematics for both of these reasons, and although I cannot promise that you will learn "all of trigonometry" from just a 30-minute lecture, I hope to at least give you the "big ideas" to enable you to see the relevance and elegance and sometimes just plain silliness that come from each of these topics.

Outline

I. To many people, the phrase "joy of mathematics" sounds like a contradiction in terms. For me, however, there are many reasons to love mathematics, which I sum up as the ABCs: You can love mathematics for its applications, for its beauty (and structure), and for its certainty.

II. What are some of the applications of mathematics?

- **A.** Mathematics is the language of science. The laws of nature, in particular, are written in calculus and differential equations.
  - **1.** Calculus tells us how things change and grow over time, modeling everything from the motion of pendulums to galaxies.
  - **2.** On a more down-to-Earth level, mathematics can be used to model how your money grows. We'll talk about the mathematics of compound interest and how it connects to the mysterious number *e*.
- **B.** Mathematics can bring order to your life. As an example, consider the number of ways you could arrange eight books on a bookshelf. Believe it or not, if you arranged the books in a different order every day, you would need 40,320 days to arrange them in all possible orders!

III. Mathematics is often taught as an exercise in disciplined thinking; if you don't make any mistakes, you'll always end up with the same answer.

A. In this way, mathematics can train people to follow directions carefully, but mathematics should also be used as an opportunity for creative thinking. One of the life lessons that people can learn from mathematics is that problems can be solved in several ways.

B. As a child, I remember thinking about the numbers that add up to 20; specifically, I wondered what two numbers that add up to 20 would have the greatest product.

1. The result of multiplying 10×10 is 100; could two other numbers that add up to 20 have a greater product?
2. I tried various combinations, such as 9×11, 8×12, 7×13, 6×14, and so on. For 9×11, the answer is 99, just 1 shy of 100. For 8×12, the answer is 96, 4 shy of 100.
3. As I continued, I noticed two things. First, the products of those numbers get progressively smaller, but more interesting is the fact that the result of each multiplication is a perfect square away from 100. In other words, 9×11 is 99, or 1 (1^2), away from 100; 8×12 is 96, or 4 (2^2), away from 100; and so on.
4. I then tried the same experiment with numbers that add up to 26. Starting with $13\times 13 = 169$, I found that $12\times 14 = 168$, just shy of 169 by 1. The next combination was $11\times 15 = 165$, shy of 169 by 4, and the pattern continued.
5. I also found that I could put this pattern to use. If I had the multiplication problem 13×13, I could substitute an easier problem, 10×16, and adjust my answer by adding 9. Because 10 and 16 are each 3 away from 13, all I had to do was add 3^2, which is 9, to arrive at the correct answer for 13×13, 169.

C. In this course, we'll go into more detail about how to square numbers and multiply numbers in your head faster than you ever thought possible. Let's look at one more example here.

1. Let's multiply two numbers that are close to 100, such as 104 and 109. The first number, 104, is 4 away from 100, and the second, 109, is 9 away from 100.
2. The first step is to add 104 + 9 (or 109 + 4) to arrive at 113 and keep that answer in mind. Next, multiply the two single-digit numbers, $4\times 9 = 36$.
3. Believe it or not, you now have the answer to 104×109, which is 11,336. We'll see why that works later in this course.

D. Another creative use of mathematics is in games. By understanding such areas of math as probability and *combinatorics* (clever ways of counting things), you can become a better game player. In this course, we'll use math to analyze poker, roulette, and craps.

IV. Throughout these lectures, you'll be exposed to ideas from high school and college-level mathematics all the way to unsolved problems in mathematics.

A. You'll learn the fundamental theorem of arithmetic, the fundamental theorem of algebra, and even the fundamental theorem of calculus.

B. Along the way, we'll encounter some of the great historical figures in mathematics, such as Euclid, Gauss, and Euler.

C. You'll learn why 0.999999… going on forever is actually equal to the number 1. It's not just close to 1, but equal to it.

D. You will actually understand why $\sin^2+\cos^2=1$, and you'll be able to follow the proof of the Pythagorean theorem and know why it's true.

V. As I said earlier, the B in the ABCs of loving mathematics is its beauty.

A. We'll study some of the beautiful numbers in mathematics, such as *e*, pi (π), and *i*. We'll see that *e* is the most important number in calculus. Pi, of course, is the most important number in geometry and trigonometry. And *i* is the imaginary number, whose square is equal to -1.

B. We'll also look at some beautiful and useful mathematical formulas, such as $e^{\pi i}+1=0$; that one equation uses *e*, pi, *i*, 1, and 0—arguably the five most important numbers in mathematics—along with addition, multiplication, exponentiation, and equality.

C. Another beautiful aspect of mathematics is patterns. In fact, mathematics is the science of patterns. We'll have an entire lecture devoted to Pascal's triangle, but let's take a look at it here, because it contains many beautiful patterns.

1. Pascal's triangle has 1s along the borders and other numbers in the middle. The numbers in the middle are created by adding two adjacent numbers and writing their total underneath.
2. We can find one pattern in this triangle if we add the numbers across each row. The results are all powers of 2: 1, 2, 4, 8, 16….
3. The diagonal sums in this triangle are all *Fibonacci numbers*: 1, 1, 2, 3, 5, 8, 13…. We'll discuss these mysterious numbers in detail.

D. Perhaps nothing is more intriguing in mathematics than the notion of infinity. We'll study infinity, both as a number-like object and as the size of an object. We'll see that in some cases, one set with an infinite number of objects may be substantially more infinite than another set with an infinite number of objects. There are actually different levels of infinity that have many beautiful and practical applications.

1. We'll also have some fun adding up infinitely many numbers. We'll see two ways of showing that the sum of a series of fractions whose denominators are powers of 2, such as 1 + 1/2 + 1/4 + 1/8 + 1/16 + …, is equal to 2. We'll see this result both from a visual perspective and from an algebraic perspective.

2. Paradoxically, we'll look at a simpler set of numbers, 1 + 1/2 + 1/3 + 1/4 + 1/5 + … (called the *harmonic series*), and we'll see that even though the terms are getting smaller and closer to 0, in this case, the sum of those numbers is actually infinite.
3. In fact, we'll encounter many paradoxes once we enter the land of infinity. We'll find a collection of numbers, an infinite collection of numbers, such that when we rearrange the numbers, we get a different sum. In other words, when we add an infinite number of numbers, we'll see that the commutative law of addition can actually fail.

E. Another problem we'll explore in this course has to do with birthdays. How many people would you need to invite to a party to have a 50% chance that two people will share the same birth month and day? Would you believe that the answer is just 23 people?

VI. The C in the ABCs of loving mathematics is certainty. In no other discipline can we show things to be absolutely, unmistakably true.

A. For example, the Pythagorean theorem is just as true today as it was thousands of years ago.

B. Not only can you prove things with absolute certainty in mathematics, but you can also prove that certain things are impossible. We'll prove, for example, that $\sqrt{2}$ is irrational, meaning that it cannot be written as a fraction with an integer (a whole number) in both the numerator and the denominator.

VII. Keep in mind that you can skip around in these lectures or view certain lectures again. In fact, some of these lectures may actually make more sense to you after you've gone beyond them, then come back to revisit them.

VIII. What are the broad areas that we'll cover in this course?

A. We'll start with the joy of numbers, the joy of primes, the joy of counting, and the joy of the Fibonacci numbers.

B. We'll have a few lectures about the joy of algebra because that's one of the most useful mathematics courses beyond arithmetic.

C. We'll talk a little bit about the joy of 9 before we turn to the joy of proofs, geometry, and the most important number in geometry, pi.

D. We'll learn about trigonometry, including sines, cosines, tangents, triangles, and circles, and we'll learn about the joy of the imaginary number i, whose square is –1. After i, we'll learn about the number e.

E. We'll talk about the joy of infinity and the joy of infinite series, setting the stage for three lectures on the joy of calculus.

F. After that, we'll study the glory of Pascal's triangle and apply some of the ideas we've learned to the joy of probability and the joy of games.

G. We'll end the course with a mathematical magic show. And that's one more reason to love math: Mathematics is truly magical.

IX. We'll close with a mathematics analogy: Later in the course, we'll see the trigonometric function, the sine function, in three different ways.

A. We'll see, for instance, in terms of a right triangle, that the sine of an angle a is equal to the length of the opposite side divided by the length of the hypotenuse.

B. We'll think of the sine function in terms of the unit circle that is given in angle a; the sine will be the y coordinate of the point (x,y) on the unit circle corresponding to that angle.

C. We'll even look at the sine function from an algebraic standpoint. That is, we'll be able to calculate the sine of a. For any angle a written in radians, we can write $\sin a = a - \frac{a^3}{3!} + \frac{a^5}{5!} + \frac{a^7}{7!} + \ldots$. We can write the sine function as an infinite sum.

D. You may think that you will never benefit from this discussion of the sine function, but mathematics is food for the brain. It can teach you how to think precisely, decisively, and creatively, even if you never use a trigonometric function.

E. Math helps you to look at the world in a different way, whether you use it to quantify decisions in daily life or you come to appreciate a fine proof in the same way that other people appreciate great poetry, painting, music, and fine wine. I invite you now to join me as we explore the joy of math together.

Reading:

William Dunham, *The Mathematical Universe: An Alphabetical Journey through the Great Proofs, Problems, and Personalities*.

Martin Gardner, *Martin Gardner's Mathematical Games*.

Martin Gardner, *Aha!: Aha! Insight and Aha! Gotcha*.

John Allen Paulos, *A Mathematician Reads the Newspaper*.

Math Horizons magazine.

Wolfram Mathworld, mathworld.wolfram.com/.

Questions to Consider:

1. When you think of your experience with mathematics, what adjectives come to mind? What experiences were unpleasant? Were there any experiences that you would describe as joyful?
2. Think of all the places where math intersects your life on a daily basis. For instance, where do you encounter math when reading the daily paper?

Lecture Two

The Joy of Numbers

Scope: In this lecture, we'll begin our joyful journey through mathematics with the joy of numbers. We'll look at the base-10 number system that we mostly use today, as well as the base-8 system; the base-2, or binary system, used in computers; and the hexadecimal system. We'll also meet Carl Friedrich Gauss, who figured out a quick way to add the numbers 1 through 100 when he was only 9 or 10 years old. We'll explore the patterns we can find in triangular numbers, prove the distributive property, and close by learning a trick for adding all the numbers in a 10-by-10 multiplication table.

Outline

I. In everyday mathematics, we use the base-10 number system, also called the *Hindu-Arabic system*.

   A. Before this system came into use, quantities had to be represented in a one-to-one correspondence. For instance, if you wanted to represent 23 animals, you'd have to line up 23 stones.

      1. In the base-10 system, we think of a number such as 23 as two rows or groups of 10, followed by a group of 3: $10 + 10 + 3 = 23$. A larger number, such as 347, is represented as three groups of 100, four groups of 10, and 7 individual units.
      2. The number 0 plays an important role as a placeholder in this system. For instance, 106 would be represented as one group of 100 plus 6 individual units, with zero 10s.

   B. In some areas of science and mathematics, other systems are used, such as base 8.

      1. For a number such as 12, instead of counting in groups of 10, we count in groups of 8. Thus, the number 12 would be written as 14 (base 8), and 23 would be written as 27 (base 8).
      2. If we were counting 12 stones in base 10, we would group the stones in one row of 10, followed by one row of 2. In base 8, we group the stones in one row of 8 and one row of 4.
      3. If we were counting 63 in base 8, we would group the stones in seven rows of 8 and one row of 7; thus, the number 63 would be written as 77 (base 8).
      4. What if we were counting 64 stones? Would we group the stones in eight rows of 8? The answer is no, because when we're working in base 8, we have only the digits 0 through 7. Instead, we have to group the stones in one large block of 64, and the number would be written as 100 (base 8).

      5. A number such as 347 would have five blocks of 8, plus three rows of 8, plus 3 individual units. This would be written as 533 (base 8).

   C. The binary number system, base 2, is used constantly in computers.

      1. In this system, we work in powers of 2, and for any number, we write a 1 every time we have a power of 2. The number 12, for example, is $1\times 2^3 + 1\times 2^2 + 0\times 2 + 0\times 1$. Substituting a 1 for each power of 2, we get 1100 (base 2).
      2. The number 64 is a power of 2 by itself, with nothing left over. It would be written as a 1 for the 64, but a 0 for the 32, 16, 8, 4, 2, and 1, or 1000000 (base 2).

   D. The hexadecimal system is also used frequently in computers.

      1. In this system, instead of having ten digits, 0 through 9, or two digits, 0 and 1, we have sixteen digits. These are the digits 0 through 9, along with A, B, C, D, E, and F, which represent the numbers 10, 11, 12, 13, 14, and 15.
      2. In base 10, the number 42 would be represented as four 10s and one 2, but in a hexadecimal system, 42 would be four 16s and one 2, or $4\times 16 + 2\times 1 = 66$ (base 16).
      3. In the hexadecimal number 2B4, the 2 represents two 16^2s and the B represents eleven groups of 16, plus 4 units. When you add all those numbers together, you get the number 692 in base 10.
      4. Let's look at the hexadecimal number FADE. This translates to $\mathrm{F}\times 16^3 + \mathrm{A}\times 16^2 + \mathrm{D}\times 16 + \mathrm{E}\times 1$, which when written in terms of base-10 numbers is $15\times 16^3 + 10\times 16^2 + 13\times 16 + 14\times 1 = 64{,}222$.
      5. Here's a trick question: What would 190 in hexadecimal be? This number would be written as $11\times 16 + 14$, or $\mathrm{B}\times 16 + \mathrm{E}$. The answer is BE, the last word of my question.

II. Let's turn now to a concept that we take for granted, but one that took thousands of years for people to figure out: the idea of negative numbers.

   A. Imagine trying to convince someone that there are numbers that are less than 0. How can something be less than nothing?

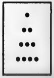
   B. But if we think of numbers as lying on a number line, with positive numbers on the right of 0 and negative numbers on the left of 0, then negative numbers don't appear to be as mysterious.

III. Carl Friedrich Gauss (1777–1855) was a great mathematician who seems to have been a genius from a young age.

   A. According to one story, when Carl was only 9 or 10 years old, his teacher asked the class to add the numbers 1 through 100. Young Gauss immediately gave the correct answer, 5,050.

B. What Gauss did was to think of the numbers as two groups, 1 through 50 and 51 through 100. He then added those numbers in pairs: $1+100=101$, $2+99=101$, $3+98=101$, … $50+51=101$. The result is fifty 101s, or $50\times 101=5{,}050$.

C. We call numbers like 5,050 *triangular numbers*, that is, numbers that can be represented in a triangle. Think of rows of 1 dot, 2 dots, 3 dots, and so on, making a triangle.



1. Using this picture of triangular numbers, we can see another way to come up with a formula for the n^{th} triangular number, that is, another formula for the sum of the first n numbers.
2. For example, imagine I put together two triangles of the same shape to create a rectangle. I use 1 + 2 + 3 + 4 dots in the form of a triangle and invert another triangle of 1 + 2 + 3 + 4 dots.
3. How many dots are in the rectangle? The rectangle has four rows of 5 dots, which means that if I added 1 + 2 + 3 + 4 twice, I would have 20 dots. If I cut that number in half, I'll have 10 dots for the number that was in a single triangle.
4. If we use one triangle with 1 + 2 + 3 up through n dots and we invert another triangle to create a rectangle that has n rows of $n+1$ dots, then the n^{th} triangular number plus the n^{th} triangular number is $n(n+1)$. In other words, the n^{th} triangular number is $\frac{n(n+1)}{2}$.
5. What is the sum of the first n even numbers, 2 + 4 + 6, all the way up to the number $2n$? Let's reduce this to a problem that we already know how to solve. We can factor out a 2 from each of those terms, which would leave us with 2 times the quantity $(1 + 2 + 3 + \ldots + n)$. We already know that sum is $\frac{n(n+1)}{2}$. Multiplying the sum by 2, the 2s cancel out, and the answer is $n(n+1)$.
6. What is the sum of the first n odd numbers? The first odd number is 1, followed by 3, and $1+3=4$ (or 2^2); the next odd number is 5, and $1+3+5=9$ (or 3^2). Continuing, we start to see a pattern: The sum of the first n odd numbers is n^2.
7. Why is the sum of the first five odd numbers 5^2? If we imagine a square divided into five rows of five squares, and we examine those squares one layer at a time, we can see why the sum of the first n odd numbers is exactly n^2.

D. Let's look at one more pretty pattern involving odd numbers.

1. We start by adding one odd number, then the next two odd numbers, then the next three odd numbers, and so on: 1, $3+5=8$, $7+9+11=27$, $13+15+17+19=64$, $21+23+25+27+29=125$. The sums here are all cubes: 1^3, 2^3, 3^3, 4^3, 5^3.

2. We next add these cubes: $1^3=1$, $1^3+2^3=9$, $1^3+2^3+3^3=36$, $1^3+2^3+3^3+4^3=100$, $1^3+2^3+3^3+4^3+5^3=225$. Those sums are perfect squares: 1^2, 3^2, 6^2, 10^2, 15^2, but they're not just any perfect squares—they're the perfect squares of the triangular numbers.
3. In other words, the sum of the cubes of the first five numbers is equal to the first five numbers summed, then squared. We can also say: The sum of the cubes is the square of the sum. Does this pattern hold for all numbers? As we'll see, the answer is yes.

IV. Let's look now at the patterns in the multiplication table.

A. We'll start by asking a question that you probably haven't thought about since you were in elementary school: Why is 3×5 equal to 5×3?

B. Let's look at a picture with dots and count the dots. You might see three rows of five dots in the picture (3×5), or you might see five columns of three dots (5×3). Because both answers are right, they both must represent the same quantity, which is why 3×5 is the same as 5×3.

C. Here's another question you probably haven't thought about in a long time: Why should a negative number multiplied by a negative number equal a positive? The answer is based on the *distributive law* in mathematics, which says: $a(b+c)=(a\times b)+(a\times c)$.

1. Imagine I have a bags of coins, and each bag of coins has b silver coins and c copper coins. Every bag has $(b+c)$ coins, and I have a bags; therefore, the total number of coins is $a(b+c)$.
2. It's also true that I have $(a\times b)$ silver coins (because each bag has b silver coins in it) and $(a\times c)$ copper coins (because each bag has c copper coins). The total number of coins, then, is $(a\times b)+(a\times c)$.
3. Both answers are right, and therefore, they're equal. We can see that the distributive law works when all the numbers are positive.

D. The distributive law should also work when the numbers are negative.

1. Let's start with an obvious statement: $-3\times 0=0$. We can replace 0 here with $(-5+5)$ because that is also equal to 0.
2. If we want the distributive law to work for negative numbers, then it should be true that $(-3\times -5)+(-3\times 5)=0$. We know that $-3\times 5=-15$, which leaves us with $(-3\times -5)-15=0$. We also know that 15 is the only number that results in 0 when we subtract it from 15; that's why $-3\times -5=+15$.

E. Let's look at one last question in this lecture: Can you add up all the numbers in a 10-by-10 multiplication table?

1. You'll need one skill to answer this question: how to square any number that ends in 5. First of all, if you square a number that ends in 5, the answer will always end in 25, such as 35^2, which equals 1,225. To find how the answer begins, multiply the first digit of

the original number, in this case 3, by the next higher digit, in this case 4: $3 \times 4 = 12$, so the answer is 1,225.

2. The original question was: What's the sum of all the numbers in the multiplication table? The numbers in the first row are 1 through 10, the sum of which is 55. That's five pairs of 11, or the triangular number formula $\frac{10 \times 11}{2} = 55$.
3. The second row of the multiplication table has the numbers 2, 4, 6, 8, 10, and so on, which is twice the sum of the numbers 1, 2, 3, through 10. Thus, that row will add up to 2×55.
4. The third row will add up to 3×55, because it's $3 \times (1 + 2 + 3 + 4 + \ldots + 10)$. The fourth row will add up to 4×55; we can continue to the tenth row, which will add up to 10×55.
5. If we were to add up all the numbers in the multiplication table, then, we would have $(1 \times 55) + (2 \times 55) + (3 \times 55) + \ldots + (10 \times 55)$. By the distributive law, that's $(1 + 2 + 3 + 4 + 5 + \ldots + 10) \times 55$, and we know the sum of the numbers 1 through 10 equals 55. Thus, if you were to sum all the numbers in the multiplication table, the answer would be 55×55.
6. Returning to the trick we learned earlier, we know that the answer to 55×55 will end in 25 and begin with 5×6, or 30; therefore, the sum of all the numbers in the multiplication table is 3,025.

Reading:

Arthur Benjamin and Michael Shermer, *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks*.

Edward B. Burger and Michael Starbird, *The Heart of Mathematics: An Invitation to Effective Thinking*, chapter 2.

John H. Conway and Richard K. Guy, *The Book of Numbers*.

Benedict Gross and Joe Harris, *The Magic of Numbers*.

Questions to Consider:

1. What is the sum of the numbers from 100 to 1,000? Using the formula for the sum of the first n numbers, find a formula for the sum of all the numbers between a and b, where a and b can be any two positive integers.
2. The alternating sum of the first five numbers is $1 - 2 + 3 - 4 + 5 = 3$. Find a formula for the alternating sum of the first n numbers. How about the alternating sum of the squares of the first n numbers?

Lecture Three
The Joy of Primes

Scope: The prime numbers, 2, 3, 5, 7, 11, 13, 17, 19, 23,…, are the building blocks of our number system. Every positive number can be written as the product of primes in exactly one way. As we examine larger numbers, prime numbers become rarer; we can even find a million consecutive composite numbers. Nevertheless, there are infinitely many primes. Many facts about prime numbers were known and proved by the ancient Greeks, but some simple questions about primes remain unanswered. Prime numbers play a significant role in theoretical mathematics, as well as in day-to-day computer security issues. You could say that understanding their properties is of prime importance to many people!

Outline

I. Prime numbers, as we'll see, are the building blocks for all the integers around us. We'll restrict our attention in this lecture to positive integers.

   A. Let's start by asking a simple question: Which numbers divide evenly into the number 12? The divisors, or *factors*, of 12 are 1, 2, 3, 4, 6, and 12. Similarly, the divisors of the number 30 are 1, 2, 3, 5, 6, 10, 15, and 30. Both of those lists have some divisors in common, namely 1, 2, 3, and 6, and the greatest common divisor (GCD) of those numbers is 6.

   B. Here's a clever way of calculating the GCD of two numbers:

      1. We want to find GCD (1,323, 896). Any number that divides evenly into 1,323 and 896 must divide evenly into 896 and into 1,323 – 896. In other words, if a number divides both x and y, then that number must also divide $x - y$. Thus, anything that divides 1,323 and 896 must also divide their difference, 1,323 – 896.
      2. How would you do that subtraction in your head? Subtract 900 from 1,323 (= 423), then add 4 back in to get 427. We might also say that any number that divides 896 and 427 must divide their sum, 896 + 427, which is 1,323.
      3. In summary, we've shown that any number that divides the first two numbers, 1,323 and 896, will also divide the next two numbers, 896 and 427. In particular, the greatest of those numbers, the GCD, must be the same. That is, GCD (1,323, 896) = GCD (896, 427). We have now replaced a large number, 1,323, with a smaller number, 427.
      4. This idea goes back to Euclid, an ancient Greek geometer. According to Euclid's algorithm: To find GCD (n, m), divide n by

m; the result will be a quotient and a remainder. If $n = qm + r$, then GCD (n, m) will be the same as GCD (m, r).

5. Let's return to the problem of finding GCD (896, 427). When you divide 427 into 896, you get a quotient of 2 and a remainder of 42, which also means that $896 = 2(427) + 42$. Euclid's algorithm tells us that GCD (896, 427) is the same as GCD (427, 42). Next, 427 divided by 42 is 10 with a remainder of 7, which means that we can simplify GCD (427, 42) to GCD (42, 7).
6. These numbers are now small enough to work with, and we can see that the greatest number that divides evenly into 42 and 7 is 7 itself. Therefore, the GCD of the original two numbers, 1,323 and 896, was shown to be the GCD of the next two numbers, 896 and 427, which was shown to be the GCD of the next two numbers, all the way down to 7, which is the GCD of the first two numbers.

II. Let's turn now to prime numbers.

A. A number is *prime* if it has exactly two divisors, 1 and itself. A number is *composite* if it has three or more divisors. Both prime and composite numbers must be positive; the number 1 is neither prime nor composite.

B. As I said, the prime numbers are the building blocks of all the integers, and this idea is expressed in what's called the *fundamental theorem of arithmetic*, or the *unique factorization theorem*.

1. According to this theorem, every number greater than 1 can be written as the product of primes in exactly one way. As an example, let's look at 5,600. To factor that number, we might say that 56 is 8×7; thus, 5,600 is $8\times 7\times 10\times 10$. That's a factorization, but it's not a prime factorization.
2. The number 8 can be broken into prime factors $2\times 2\times 2$; the number 7 is already prime. Both 10s can be factored as 2×5. If we put these together, the prime factorization of 5,600 is $2^5\times 5^2\times 7^1$.
3. How many divisors does 5,600 have? For a number to be a divisor of 5,600, its only prime factors could be 2, 5, and 7.
4. What's the largest power of 2 that could be a divisor of 5,600? The answer is 5, because if we used 2 to a higher power, such as 2^6 or 2^7, then the result wouldn't divide into 5,600.
5. Any divisor of 5,600 will have to be of the form $2^a\times 5^b\times 7^c$, where a could be as small as 0 or as high as 5, b could be as small as 0 or as high as 2, and c could be as small as 0 or as high as 1.
6. If we let $a = 3$, $b = 1$, and $c = 0$, then the divisor would be $2^3\times 5^1\times 7^0$, or $8\times 5\times 1$, which is 40. What if we chose all 0s? Then, the divisor would be $2^0\times 5^0\times 7^0$, or $1\times 1\times 1$, which is 1.
7. The answer to the question of how many divisors 5,600 has will be in the form of $2^a\times 5^b\times 7^c$. We have six possibilities for a, any number between 0 and 5; three possibilities for b; and two possibilities for c. Therefore, 5,600 has $6\times 3\times 2$, or 36 divisors.

III. Let's look at the complementary idea of least common multiples (LCMs).

A. We look at 12 and 30 again. The number 12 has multiples 12, 24, 36, 48, 60, 72, …. The number 30 has multiples 30, 60, 90, 120, 150, 180,…. Comparing those lists, 60 is the smallest multiple of 12 and 30.

B. Recall that GCD (12, 30) = 6, and if we multiply 6×60, we get 360. If we multiply 12×30, we also get 360. That's a consequence of the theorem that for any numbers a and b, GCD $(a, b)\times$ LCM $(a, b) = ab$.

IV. A concept we will use frequently in these lectures is *factorials*.

A. The number $n!$ (n factorial) is defined to be $n\times (n-1)\times (n-2)\times \ldots$ down to 1. For example, 3! is $3\times 2\times 1$, which is 6; 4! is $4\times 3\times 2\times 1$, which is 24. Notice that $n!$ can be defined recursively as $n\times (n-1)!$.

B. I claim that 0! is 1, partly because if I want the equation $n! = n\times (n-1)!$ to be true, I need 0! to be 1. Let's look at this idea: $1! = 1\times 0!$, and $1! = 1$. If we want $1\times 0!$ to be 1, then 0! must be defined as 1.

C. The number 10! is 3,628,800, but that pales in comparison to 100!. How many 0s will be at the end of the number 100!?

1. The number 100! has a prime factorization that can be written in the form $2^a\times 3^b\times 5^c\times 7^d\ldots$ To find how many 0s are at the end of that number, we have to ask ourselves how 0s are made.
2. Every time 2 and 5 are multiplied, the result is a 10, which creates a 0 at the end of a number.
3. For 100!, the only numbers that matter in terms of creating 0s are the power of 2 and the power of 5. The smaller of those numbers, a or c, will be the number of 0s in the result of 100!.
4. Looking at 100!, we see that there will be more powers of 2 in its factorization, so the smaller exponent, the one we're interested in, is the power of 5, the exponent of c. The number of 0s at the end of 100! will be this exponent.
5. How do we find the number of 5s in the factorization of 100!? There are 20 multiples of 5 in the numbers 1 through 100, and each contributes a factor of 5 to the prime factorization of 100!.
6. Keep in mind that some of those multiples of 5 (namely, 25, 50, 75, and 100) will each contribute an extra factor of 5. Thus, the total contribution of 5s to 100! is 20 + 4, or 24.
7. With 200!, there are 40 multiples of 5, each contributing a 5 to the 200!. All the multiples of 25 each contribute an extra factor of 5, and there are 8 of those up to 200. Finally, the number 125 is 5^3. or $5\times 5\times 5$, so it contributes one more factor of 5. Thus, the total number of 0s at the end of 200! will be 49.
8. The number 100! is 9.3×10^{157}, which has 24 zeros on the end.

V. Let's return to prime numbers.

A. As we look at larger numbers, the primes become a bit scarcer because there are more numbers beneath them that could possibly divide them.

B. Do the primes ever die out completely? Is there a point after which every number is composite? It seems possible, yet we can prove that the number of primes is, in fact, infinite.

1. Suppose that there were only a finite number of primes. That would mean that there would have to be some prime number that was bigger than all the other prime numbers. Let's call that number P. Every number, then, would be divisible by 2, 3, 5, … or P.
2. Now, let's look at the number $P!$. This number, $P!$, will be divisible by 2, 3, 4, 5, 6, 7, … and every number through P, because it is equal to the product of all those things.
3. Next, let's look at the number $P! + 1$. Can 2 divide evenly into $P! + 1$? No, because 2 divides into $P!$, and if that's true, 2 will not divide into $P! + 1$. Can 3 divide into $P! + 1$? Again, the answer is no, because 3 divides into $P!$, and thus, it can't divide into $P! + 1$. In fact, all the numbers between 2 and P will divide $P!$; therefore, none of them will divide $P! + 1$. That contradicts our assertion that all numbers were divisible by something between 2 and P.
4. Suppose we thought 5 was the biggest prime. We know that the number $5! + 1$ will not be divisible by 2, 3, 4, or 5; therefore, 5 could not be the biggest prime. Does that mean $5! + 1$ is prime? No. In fact, $5! + 1 = 121$, which is 11^2, and 11 is a bigger prime than 5. We know that 5 is not the biggest prime because $P!+1$ will either be prime or it will be divided by a prime that's larger than P.

C. Given that there are an infinite number of primes, is it true that we have to encounter a prime every so often, or would it be possible to find, for example, 99 consecutive composite numbers? I claim the answer is yes.

1. I claim that the 99 consecutive numbers from $100! + 2$, $100! + 3$, $100! + 4$, …, $100! + 100$ are all composite. We know that $100!$ is divisible by 2, 3, 4, …, 100. And we know that since 2 divides into $100!$, it will also divide into $100! + 2$. Further, since 3 divides into $100!$, it will divide into $100! + 3$, and so on. Thus, since 100 divides into $100!$, it will divide into $100! + 100$.
2. Therefore, all those numbers are composite, because the first one is divisible by 2, the second one is divisible by 3, and so on, until the last one, which is divisible by 100.

D. The largest prime number that has been found so far, discovered in 2006 by a mathematician named Curtis Cooper, is $2^{32,582,657} - 1$. The resulting prime number is more than 9 million digits long.

VI. Let's close with a couple of more questions about prime numbers.

A. A *perfect number* is a number that's equal to the sum of all its *proper divisors* (all the divisors except the number itself). For example, 6 has proper divisors 1, 2, and 3. The next perfect numbers are 28, 496, and 8,128. Let's look at the prime factorizations of these numbers.

1. The prime factorization of 6 is 2×3; 28 is 4×7; 496 is 16×31; and 8,128 is 64×127. The first number in all these equations is a power of 2; the second number is one less than twice the original number, and it is also prime.
2. In fact, the mathematician Leonhard Euler (1707–1783) showed that if P is a prime number and if $2^P - 1$ is a prime number, then the result of multiplying $2^{P-1}(2^P-1)$ will always be perfect.
3. That's true for all the even perfect numbers, but what about odd perfect numbers? No one knows if any odd perfect numbers exist.

B. A *twin prime* is a set of two prime numbers that differ by 2, for example, 3 and 5. We have found twin primes with more than 50,000 digits, yet we don't know if there are an infinite number of twin primes.

C. According to *Goldbach's conjecture*, every even number greater than 2 is the sum of two primes: for example, $6 = 3 + 3$; $18 = 11 + 7$; $1{,}000 = 997 + 3$. This problem has been verified through the zillions, but we don't have proof that it is true for all even numbers.

1. It has been proved, by the way, that every even number is the sum of at most 300,000 primes.
2. We also have proof that with large enough numbers, we reach a point where every number is of the form $P + QR$, where P is prime and QR is almost prime, meaning that QR is a number that has at most two prime factors, Q and R.

D. Prime numbers have many applications, such as testing the performance, accuracy, and security of computers.

Reading:

Benedict Gross and Joe Harris, *The Magic of Numbers*, chapters 8–13, 23.

Paulo Ribenboim, *The New Book of Prime Number Records*, 3rd ed.

The Prime Pages, http://primes.utm.edu/.

Questions to Consider:

1. The primes 3, 5, and 7 form a *prime triplet*, three consecutive odd numbers that are all prime. Why do no other prime triplets exist?
2. To test if a number under 100 is prime, you need to test only whether it is divisible by 2, 3, 5, or 7. In general, to test if a number n is prime, we need to test only if it is divisible by prime numbers less than $\sqrt{n}$. Why is that true?

Lecture Four

The Joy of Counting

Scope: Numbers were first invented as a way to count and keep track of things. It stands to reason, then, that we can use numbers to find the answers to many interesting and important counting questions. In fact, there is an entire branch of mathematics devoted to studying counting questions, called *combinatorics*. It may surprise you to learn that there are more than 3 million ways to arrange 10 books on a bookshelf and almost 40 million ways to arrange 11 books. By applying some very simple ideas (the rule of sum and the rule of product), you can also determine your chances of winning the lottery or winning in poker.

Outline

I. Two principles apply to counting: the rule of sum and the rule of product.

A. According to the rule of sum, if I own five long-sleeved shirts and three short-sleeved shirts, then the number of shirts I can wear on any given day is $5 + 3$.

B. According to the rule of product, if I own eight shirts and five pairs of pants, then the number of possible outfits I can wear on any given day is 8×5. If I have ten ties, that would multiply the number of possibilities by a factor of 10. I'd have $8 \times 5 \times 10 = 400$ different outfits.

II. Knowing those two principles, let's start with a simple question, such as: In how many ways can we arrange a group of letters?

A. For instance, the letters A and B can be arranged in two ways—AB and BA. The letters A, B, and C can be arranged in six ways: ABC, BAC, CAB, ACB, BCA, CBA. There are 24 ways to arrange A, B, C, and D. These numbers are all factorials: $2 = 2!$, $6 = 3!$, and $24 = 4!$.

1. If we know there are six ways to arrange A, B, and C, let's figure out the number of ways to arrange A, B, C, and D. Starting with ABC, we could put D in the first, second, third, or fourth position. That will to lead to six new ways to arrange ABC and D, where A, B, and C are in their original positions.

2. With the next set of letters, ACB, there are still four places where we can insert the letter D among the original letters. For every one of those six arrangements, we can follow up with four new arrangements. Thus, the number of possibilities is $6 \times 4 = 24$, or $4!$.

B. Another way of thinking about factorials is to imagine placing five cards on a table in different arrangements. You have five choices for which card you'll put down first. After you've chosen that card, you have four choices for which card goes next. Then, you have three

choices for the next card, two choices for the next, and one choice for the last card; the total number of possibilities is $5 \times 4 \times 3 \times 2 \times 1$, or $5!$. In general, the number of ways of arranging n different objects is $n!$.

C. How many different five-digit zip codes are possible? The first digit is anything from 0 to 9; the second digit is from 0 to 9; and so on. For each of the digits, there are 10 choices; therefore, the number of possible zip codes is $10 \times 10 \times 10 \times 10 \times 10 = 10^5 = 100{,}000$.

D. How many zip codes are possible in which none of the numbers repeats? For the first digit, there are 10 choices, but for the second digit, there are only 9 choices; for the third digit, there are 8 choices; for the fourth, 7 choices; and for the fifth, 6 choices. The number of five-digit zip codes with no repeating numbers is $10 \times 9 \times 8 \times 7 \times 6$, or 30,240.

E. Let's apply this approach to horseracing. In a race with 8 horses, how many different outcomes are possible when the outcomes are as follows: one horse finishing first, another finishing second, and another finishing third? Again, there are 8 possibilities for the horse that comes in first, 7 for the horse that comes in second, and 6 for the horse that comes in third: $8 \times 7 \times 6 = 336$ possibilities for the outcome.

III. How many possible license plates are there if a license plate comes in two varieties? A type I license plate has three letters followed by three numbers. A type II license plate has two letters followed by four numbers. Because we have two types of license plates, the rule of sum will apply here.

A. How many type I license plates are possible? For each of the three letters, there are 26 choices. For each of the three numbers, there are 10 choices. Multiplying those choices, we get 17,576,000 different license plates.

B. How many type II license plates are possible? For the two letters, there are 26 choices each. For the four numbers, there are 10 choices each; altogether, $26 \times 26 \times 10^4 = 6{,}760{,}000$.

C. When we combine type I and type II license plates, the total number of possibilities is 24,336,000.

D. The branch of mathematics known as combinatorics allows us to solve problems in different ways. For example, we can actually do the license plate problem in one step instead of two.

1. The number of choices for the first letter is 26, and for the second letter, 26 also; whether the license plates are of type I or type II, there are 26×26 ways to get started.

2. The third item on the license plate could be a letter or a number. Thus, there are $26 + 10 = 36$ possibilities for the third item.

3. The remaining three items are all numbers; therefore, there are 10 possibilities for each. When we multiple those numbers together, $26\times 26\times 36\times 10\times 10\times 10$, we get, again, 24,336,000.

E. What if all the letters must be different on the license plate? In this case, there are 26 choices for the first letter and 25 choices for the second letter, but the third item could be any one of 24 letters or 10 numbers. Therefore, there are 34 possibilities for the third item. Then, because the last three items must all be numbers, there are 10 possibilities for each. Multiply those numbers together, and we get 22,100,000.

F. If all the letters and numbers must be different, then we can still solve the problem in one step, but we have to pursue a more creative strategy.

1. There are 26 choices for the first letter and 25 choices for the second letter. There are 10 choices for the fourth item, which must be a number; 9 choices for the next number; and 8 choices for the last number. The third item can be a number or a letter. There are 24 choices for the letter and 7 choices for the number; therefore, there are 31 possibilities for the third item.
2. Multiplying all those possibilities: $26\times 25\times 31\times 10\times 9\times 8 =$ 14,508,000.

IV. Let's now talk about winning the lottery.

A. California has a game called Super Lotto Plus, which is played as follows: First, you choose five numbers from 1 through 47. Next, you choose a mega number from 1 through 27. That mega number can be one of the five numbers you picked first or it can be a different number.

B. For the first step, we pick the first five Fibonacci numbers, 2, 3, 5, 8, and 13, and for the mega number, 21. In how many ways can the state pick its numbers, and which of those are the numbers we picked?

C. The state has 47 choices for its first number, 46 choices for its second, and so on, or $47\times 46\times 45\times 44\times 43$ ways of picking the first five numbers. Then, the state has 27 ways to pick the mega number. It seems like that would be the right answer, but we have overcounted.

D. The state might choose the numbers 1, 10, 20, 30, and 45 or the numbers 10, 20, 45, 30 and 1. They are the same group of five numbers, but we've counted them as different. In how many ways could we arrange those five numbers and still have the same set of five numbers? By dealing the cards earlier, we saw that there were 5! ways of arranging those numbers. Thus, to find the correct answer to this problem, we divide the original number that we came up with by 5!.

E. In other words, in this problem, we overcounted the possibilities for the numerator, then divided by the denominator to get the correct answer.

F. We saw, then, that the state has 41,416,353 ways to pick its numbers, only one of which is our group of five numbers. Therefore, our chance of winning is just $\frac{1}{41,416,353}$.

G. Incidentally, another way to express such products as $47\times 46\times 45\times 44\times 43$ is to multiply the numerator and the denominator by 42!; thus, the numerator would be 47! and the denominator would be 42!. Those quantities are the same thing, but the second form is cleaner. The number of ways to pick five different numbers out of 47 is $\frac{47!}{5!\times 42!}$.

V. In general, the number of ways to pick k objects from n objects when the order is not important is $\frac{n!}{k!\times(n-k)!}$. The notation for this is $\binom{n}{k}$.

A. How many 5-card poker hands are possible? We have 52 cards and we choose 5 of them. The order that you get the cards is not important for a game such as five-card draw. The number of ways of picking 5 out of 52 is $\binom{52}{5}$, which has the formula $\frac{52!}{5!\times 47!}$, which is 2,598,960.

B. What are the chances of being dealt a specific kind of hand in poker? For instance, what are your chances of being dealt five cards of the same suit, a flush?

1. We have four choices for the suit—spades, hearts, diamonds, or clubs. In how many ways can we pick 5 cards of the same suit, such as hearts, out of the 13 hearts in the deck? By definition, that is $\binom{13}{5}$; thus, we have $4\times\binom{13}{5} = 4\frac{13!}{5!\times 8!} = 5,148$.
2. The chances of being dealt a flush in poker would be 5,148 divided by the 2,598,960 possible different poker hands. That's about 0.2 percent; about 1 out of every 500 poker hands dealt will be a flush.

C. What are the chances of being dealt a full house in poker? A full house consists of five cards, three of one value and two of another value.

1. There are 13 choices for the value that will be triplicated and 12 choices for the value that will be duplicated. Let's say our two values are queens and sevens.
2. Next, we have to determine the possibilities for suits of those cards. How many possibilities for suits are there for the 3 queens?

The answer is $\binom{4}{3}$. That is, from the 4 queens in the deck—spade, heart, diamond, and club—choose 3 of them: $\binom{4}{3} = 4$.

3. Similarly, how many possibilities for suits are there for the 2 sevens? The answer is $\binom{4}{2} = 6$.
4. Thus, the number of possibilities for a full house is $13 \times 12 \times 4 \times 6 = 3{,}744$.

D. How many 5-card poker hands have at least 1 ace?

1. To answer this question, you might reason that you first have to choose an ace, then choose 4 other cards from the remaining 51. You have 4 choices for the first ace and $\binom{51}{4}$ ways of picking from the remaining 51 cards. The answer, then, would be $4 \times \binom{51}{4}$.
2. Unfortunately, that logic is incorrect. There is no "first ace" in the poker hand. To approach the problem by choosing an ace as the first card, then picking 4 other cards is to bring order into a problem where order does not belong.
3. The correct way to do the problem is to break it down into four cases. First, we count those poker hands with 1 ace; then, we count those hands with 2 aces; then, 3 aces; then, 4 aces. Then, we apply the rule of sum to add those hands together, as shown below.

Number of poker hands with 1 ace:	$\binom{4}{1} \times \binom{48}{4}$ *	Number of poker hands with 3 aces:	$\binom{4}{3} \times \binom{48}{2}$
Number of poker hands with 2 aces:	$\binom{4}{2} \times \binom{48}{3}$	Number of poker hands with 4 aces:	$\binom{4}{4} \times \binom{48}{1}$

*Number of ways to choose 1 ace out of 4 in the deck multiplied by the number of ways to choose 4 non-aces out of 48 in the deck.

Adding the cases together, we get 886,656 different poker hands.

E. Another approach to this problem is to find how many hands have no aces and subtract that answer from the total amount.

1. There are 4 aces in the deck and 48 non-aces, and we can choose any 5 of the non-aces. In how many ways can we choose 5 things out of 48? By definition, the answer is $\binom{48}{5}$, which is 1,712,304.
2. Once we have that value, we can subtract it from the number of possible poker hands, 2,598,960, which leaves us the same number we got before, 886,656.

F. The possibilities for counting questions in horseracing, lotteries, and poker are endless, as endless as the variations of the games themselves. What happens if we allow wild cards in the game? What if you're playing seven-card stud or Texas hold 'em or blackjack? You can apply mathematics to solving problems in all these games, but you don't want to use math to take all the fun out of games!

Reading:

Arthur T. Benjamin and Jennifer J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof.*

Benedict Gross and Joe Harris, *The Magic of Numbers*, chapters 1–4.

Alan Tucker, *Applied Combinatorics*, 5th ed.

Questions to Consider:

1. How many five-digit zip codes are palindromic (that is, read the same way backwards as forwards)?
2. In how many ways can you be dealt a straight in poker (that is, five cards with consecutive values: A2345 or 23456 or ... or 10JQKA)? In how many ways can you be dealt a flush (that is, five cards of the same suit)? Compare these numbers to the number of full houses. This explains why in poker, full houses beat flushes, which beat straights.

Lecture Five

The Joy of Fibonacci Numbers

Scope: The Fibonacci numbers, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,…, make up a sequence of numbers with many beautiful and unexpected properties. They were originally described in the first Western textbook on arithmetic using Hindu-Arabic numbers, instead of the computationally clumsier Roman numerals. The numbers arose in an exercise that had to do with counting rabbits. Since then, many mathematicians have discovered intriguing patterns among these numbers, which we will explore. For example, the sum of the squares of two consecutive Fibonacci numbers is always a Fibonacci number, and the ratio of consecutive Fibonacci numbers becomes closer to the golden ratio. These numbers also arise in nature, art, computer science, and poetry.

Outline

I. In this lecture, we'll talk about the *Fibonacci numbers*. This sequence begins with the numbers 1, 1, 2, 3, 5, 8, 13, 21, and so on. We can find the sequence by adding each number to the number that precedes it.

   A. In the 12th century, Fibonacci wrote a book called *Liber Abaci* (*The Book of Calculation)*, that was the first textbook for arithmetic in the Western world and used the Hindu-Arabic system of numbers.

   B. The Fibonacci numbers arose in one of the problems from this book that involved a scenario with imaginary rabbits that never die.

      1. We begin with one pair of rabbits in month 1. After one month, the rabbits are mature, they mate, and they produce a pair of offspring, one male and one female (month 3). After one month, those offspring mature, mate, and produce a pair of offspring, giving us two pairs of adults and one pair of babies (month 4). In month 5, we'll have three pairs of adults and two pairs of babies.

      2. How many pairs will we have in month 6? We will have all five pairs of rabbits from month 5, plus all the rabbits from month 4 will now have babies, or 5 + 3 = 8. How many rabbits will we have after 12 months? By continuing this process, you can see that we will have 144 pairs of rabbits in month 12.

II. Let's look at the Fibonacci numbers from a more mathematical standpoint.

   A. We define F_1 to be the first Fibonacci number and $F_2 = 1$ to be the second Fibonacci number. We then have what's called a *recursive equation* to find the other Fibonacci numbers; according to this equation, the n^{th} Fibonacci number $(F_n) = F_{n-1} + F_{n-2}$.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
1	1	2	3	5	8	13	21	34	55	89	144

   B. What would happen if we were to start adding all the Fibonacci numbers together? For instance, we would see:

   1 + 1 = 2
   1 + 1 + 2 = 4
   1 + 1 + 2 + 3 = 7
   1 + 1 + 2 + 3 + 5 = 12
   1 + 1 + 2 + 3 + 5 + 8 = 20…

   Do you see a pattern with those numbers—1, 2, 4, 7, 12, 20?

      1. When those numbers are written as differences (2 – 1, 3 – 1, 5 – 1, 8 – 1, 13 – 1, 21 – 1), we see that they are each one number off from the Fibonacci numbers—2, 3, 5, 8. We'll look at two explanations for why this works.

      2. The first explanation is that if the formula works in the beginning, it will keep on working. We know, for example, that 1 + 1 + 2 + 3 + 5 + 8 = 21 – 1. What will happen when we add the next Fibonacci number, 13, to that system? When we add 21 + 13, we get the next Fibonacci number, 34; further, 21 – 1 + 13 = 34 – 1, and that pattern will continue forever. This is our first example of what's called a *proof by induction.*

      3. The second explanation is a bit more direct. Let's replace the first 1 in the sequence 1 + 1 + 2 + 3 + 5 + 8 = 21 – 1 with 2 – 1. Let's then replace the second 1 with 3 – 2; the 2, with 5 – 3; and so on. We're representing each of those numbers as the difference of two Fibonacci numbers: (2 – 1) + (3 – 2) + (5 – 3) + (8 – 5) + (13 – 8) + (21 – 13).

      4. Look at what happens when we add those numbers together. Starting with (2 – 1) + (3 – 2), we get a +2 and a –2, and those 2s cancel. Then, when we add 5 – 3, the 3s cancel; when we add 8 – 5, the 5s cancel; and so on. This is called a *telescoping sum*.

      5. When the dust settles, all that's left of this sum is the 21 on the right that hasn't been canceled yet and the –1 at the beginning that never got canceled. Thus, when we add all those numbers together, we get 21 – 1. The formal equation, what mathematicians call an *identity*, for $F_1 + F_2 + \ldots + F_n$ is $F_{n+2} - 1$. That is, the sum of the first n Fibonacci numbers is equal to $F_{n+2} - 1$.

   C. What would happen if we were to sum the first n even-positioned Fibonacci numbers? That is, what's $F_2 + F_4 + F_6 + \ldots + F_{2n}$?

      1. Let's begin by looking at the data: F_2 is 1, F_4 is 3, F_6 is 8, and as we add these numbers up, we have 1, 1 + 3 = 4, 1 + 3 + 8 = 12, and 1 + 3 + 8 + 21 = 33. Do you see the pattern?

      2. Rewriting those numbers, we have 2 – 1, 5 – 1, 13 – 1, 34 – 1; those differences are 2, 5, 13, 34—every other Fibonacci number.

Thus, the pattern is $F_3 - 1$, $F_5 - 1$, $F_7 - 1$, and $F_9 - 1$. Let's see why that works.

3. Look at the equation: 1 + 3 + 8 + 21. We leave the 1 alone, but we replace 3 with 1 + 2; we replace 8 with 3 + 5; and we replace 21 with 8 + 13: (1) + (1 + 2) + (3 + 5) + (8 + 13) = 34 – 1. We're adding every other Fibonacci number, and what we really have is 1 + 1 + 2 + 3 + 5 + 8 + 13, which is exactly the same pattern that we had before. The result, then, is 34 – 1, just as we saw before.

D. What would happen if we were to sum the odd-positioned Fibonacci numbers? That is, what's $F_1 + F_3 + F_5 + \ldots + F_{2n-1}$?

1. We start with 1, then 1 + 2, then 1 + 2 + 5, then 1 + 2 + 5 + 13. We see the numbers 1, 3, 8, and 21. Those are the Fibonacci numbers themselves, not disguised at all. Why does that work?
2. As before, we leave the 1 alone, but we replace 2 with 1 + 1, 5 with 2 + 3, and 13 with 5 + 8. When we add all those together, we have the same Fibonacci sum, except we have an extra 1 at the beginning; that extra 1 will cancel the –1, leaving us with an answer of 21. The formula is $F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}$.

E. Let's now look at a different pattern. Which Fibonacci numbers are even? According to the data, every third Fibonacci number appears to be even. Will this pattern continue?

1. Think about the fact that the Fibonacci numbers start off as odd, odd, even. When we add an odd number to an even number, we get an odd number. Then, when we add the even number to the next odd number, we get another odd number. When we add that odd number to the next odd number, we get an even number, and we're back to where we started: odd, odd, even. That proves that every third Fibonacci number will be even. What's more, anything that isn't a third Fibonacci number won't be even; it will be odd.
2. What if we look at every fourth Fibonacci number? Believe it or not, every fourth Fibonacci number is a multiple of 3: 3, 21, 144. Moreover, the only multiples of 3 among the Fibonacci numbers occur as every fourth Fibonacci number.
3. Every fifth Fibonacci number is a multiple of 5. Every sixth Fibonacci number is a multiple of 8, and the only multiples of 8 are F_6, F_{12}, F_{18}, F_{24}.... This theorem reads: The number F_m divides F_n if and only if m divides n.
4. Forgetting about Fibonacci numbers for just one second, what is the largest number that divides 70 and 90? In other words, what is the greatest common divisor (GCD) of 70 and 90? The answer is 10. Now, what is the largest number that divides F_{70} and F_{90}, the 70th Fibonacci number and the 90th Fibonacci number? Believe it or not, the answer is the 10th Fibonacci number.
5. In general, GCD (F_m, F_n) is always a Fibonacci number, and it's not just any Fibonacci number, but it's the most poetic Fibonacci number you could ask for. That is to say, GCD $(F_m, F_n) = F_{\text{GCD}(m,n)}$.

F. Which Fibonacci numbers are prime?

1. Looking at our list of Fibonacci numbers, we have 2, 3, 5, 13, and 89. It turns out that the first few prime Fibonacci numbers are F_3, F_4, F_5, F_7, F_{11}, F_{13}, F_{17}.... There's a pattern there; except for F_4, which we'll ignore, it looks like we're seeing prime indices.

F_3	F_4	F_5	F_7	F_{11}	F_{13}	F_{17}
2	3	5	13	89	233	1597

2. In fact, if the index is composite (except for F_4, which is a special case because it's 2×2, and F_2 and F_1 are both 1), if m is composite, then F_m is guaranteed to be composite. That's a consequence of the theorem that states m divides n if and only if F_m divides F_n.
3. Is it true that every prime index produces a prime Fibonacci number? As is often the case with prime numbers, the answer to that question is hard to pin down. If we go just a little farther out in the sequence to F_{19}, we see that 19 is prime, but F_{19} is 4,181, which is not prime; it can be factored into 113×37.
4. The only places where we see primes along the Fibonacci trail are at the Fibonacci indices. In fact, an unsolved problem in math is: Are there infinitely many prime Fibonacci numbers?
5. Even though we don't know if there are an infinite number of prime Fibonacci numbers, we do know that every prime divides a Fibonacci number. In fact, if P ends in 1 or 9, then P divides F_{P-1}. If P ends in 3 or 7, then P divides F_{P+1}. For instance, 7 divides F_8, 21, and 11, which ends in 1, divides F_{10}, which is 55. Then, 13, which ends in 3, divides F_{14}, which is 377, which is 13×29.

III. We know that if we add consecutive Fibonacci numbers together, we get the next Fibonacci number; that's how Fibonacci numbers are made. Let's now look at the squares of Fibonacci numbers.

A. Starting off, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $5^2 = 25$, $8^2 = 64$, $13^2 = 169$, $21^2 = 441$, and so on. Look what happens if we add $1^2 + 1^2$; we get 2, a Fibonacci number. If we add $1^2 + 2^2$, we get 5, a Fibonacci number. If we add $2^2 + 3^2$, 4 + 9, we get 13, another Fibonacci number. In fact, it looks as if the sum of the squares of Fibonacci numbers is always a Fibonacci number. That is to say, $F_n^2 + F_{n+1}^2 = F_{2n+1}$.

B. What happens if we start adding up the sums of the squares, not of two consecutive Fibonacci numbers, but of all the Fibonacci numbers?

1. We begin with $1^2 + 1^2 = 2$, $1^2 + 1^2 + 2^2 = 6$, $1^2 + 1^2 + 2^2 + 3^2 = 15$; the sum of the squares of the first five Fibonacci numbers is 40. The sum of the squares of the first six Fibonacci numbers is 104.
2. If we look closely at the results of these additions—2, 6, 15, 40, 104—we see that the Fibonacci numbers are buried inside them.
3. For example, 2 is 1×2, 6 is 2×3, 15 is 3×5, 40 is 5×8, and 104 is 8×13. In fact, in general, $F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n \times F_{n+1}$.

C. Let's focus on one of the Fibonacci numbers, say, F_4, which is 3.

1. If we multiply its neighbors, F_3 and F_5, we see that 2×5 is 10, which is 1 away from 9, or F_3^2. If look at F_5, which is 5, and multiply its neighbors, 3×8, the result is 24, or 1 away from 25. Do you see the pattern?
2. This pattern works even with the lower numbers; $F_1 \times F_3$ is 1 away from 1^2, and $F_2 \times F_4$ is 1 away from F_3^2. In general, the pattern seems to be as follows: $F_{n-1} \times F_{n+1} = F_n^2 \pm 1$. In fact, we can say it more precisely: $F_{n-1} \times F_{n+1} - F_n^2 = (-1)^n$.

D. What if we look at the neighbors that are two away from a given Fibonacci number?

1. We begin with F_3, which is 2, and multiply its neighbors two to the left and two to the right. We get $1 \times 5 = 5$, which is 1 away from 4, or 2^2.
2. Let's try the same thing with F_4, which is 3. We multiply its two-away neighbors, 1×8, which is 8, or 1 away from 9, or 3^2. The general pattern is $F_{n-2} \times F_{n+2} - F_n^2 = (-1)^n$.

E. If we look three away from a given Fibonacci number, we see the same sort of pattern. Looking at F_5, 5, and multiplying its three-away neighbors, we get $1 \times 21 = 21$, which is 4 away from 25. Looking at F_6, 8, and multiplying its three-away neighbors, we get $2 \times 34 = 68$, which is 4 away from 64. The differences between these neighboring multiplications are 1, 1, 4, 9, 25, 64—squares of the Fibonacci numbers.

IV. Let's now turn to some division properties of Fibonacci numbers.

A. The ratios of consecutive Fibonacci numbers (shown at right) seem to converge on what's known as the *golden ratio*.

$\frac{1}{1} = 1$	$\frac{8}{5} = 1.6$
$\frac{2}{1} = 2$	$\frac{13}{8} = 1.625$
$\frac{3}{2} = 1.5$	$\frac{21}{13} = 1.615\ldots$
$\frac{5}{3} = 1.666\ldots$	
golden ratio: $\frac{1+\sqrt{5}}{2} = 1.618\ldots.$	

B. Let's look briefly at the properties of the golden ratio.

1. We start with a rectangle of dimensions 1 and 1.618… and cut out a 1-by-1 square, leaving a rectangle with height of 1 and length of .618…. Rotating the second rectangle 90 degrees, we have a rectangle that is proportional to the first, with height of .618… and length of 1.
2. Thus, the ratio of $\frac{1.618\ldots}{1}$ is the same as the ratio of $\frac{1}{.618\ldots}$.

C. Here's another connection between the golden ratio and the Fibonacci numbers, known as *Binet's formula.* Amazingly, this formula, shown at right, produces the Fibonacci numbers, and it can be used to explain many of the Fibonacci numbers' beautiful properties.

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

D. The Fibonacci numbers appear everywhere in nature, and they show up in computer science, in arts and crafts, and even in poetry.

Reading:

Ron Knott, *Fibonacci Numbers and the Golden Section*, www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html.

Thomas Koshy, *Fibonacci and Lucas Numbers with Applications*.

Mario Livio, *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number*.

Arthur T. Benjamin and Jennifer J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof.*

Fibonacci Association, www.mscs.dal.ca/Fibonacci.

Questions to Consider:

1. Investigate what you get when you sum every third Fibonacci number. How about every fourth Fibonacci number?
2. Close cousins of the Fibonacci numbers are the Lucas numbers: 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ... What patterns can you find inside this sequence?

L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}
2	1	3	4	7	11	18	29	47	76	123	199	322

For instance, what do you get when you add Lucas numbers that are two apart? What is the sum of the first n Lucas numbers? How about the sum of the squares of two consecutive Lucas numbers? What happens to the ratio of two consecutive Lucas numbers?

Lecture Six
The Joy of Algebra

Scope: Think of a number, double it, add 10, then divide by 2. Now, subtract your original number. I'll bet that your result is the number 5. Why does this work? Algebra. Algebra is probably the most important subject in mathematics because it introduces the idea of using an abstract variable to represent an unknown quantity for which we wish to solve. The subject was introduced in the 9th century by the Persian mathematician Al-Khowarizmi (from whose name we derive the word *algorithm*) in his book *Al-jabr* on the science of transposition and cancellation. In this lecture, we'll learn to allow letters, such as x or y or L or A, to represent unknown quantities that we add, subtract, multiply, and divide. We'll also apply the *golden rule of algebra* (do unto one side as you do unto the other) to solve fanciful and real-world word problems and to demystify mathematical magic tricks.

Outline

I. We begin this lecture by exploring the magic trick we started with.

A. Algebra assigns variables to unknown quantities. When I asked you to think of a number between 1 and 10, I called that unknown number n. The next step was to double that number: $n + n$, or $2n$. The next step was to add 10: $2n + 10$. Then, divide by 2: $\frac{2n+10}{2} = n+5$. Finally, subtract the original number: $n + 5 - n = 5$.

B. Let's do another trick. This time, think of two numbers between 1 and 20. Let's say you chose the numbers 9 and 2. We'll then start adding these two consecutive numbers to get the next number in a sequence, as shown in the table on the left below.

1	9
2	2
3	9 + 2 = 11
4	2 + 11 = 13
5	11 + 13 = 24
6	13 + 24 = 37
7	24 + 37 = 61
8	37 + 61 = 98
9	61 + 98 = 159
10	98 + 159 = 257

1	x
2	y
3	$x + y$
4	$x + 2y$
5	$2x + 3y$
6	$3x + 5y$
7	$5x + 8y$
8	$8x + 13y$
9	$13x + 21y$
10	$21x + 34y$

Before we continue, note that the sum of the numbers in rows 1 through 10 is 671.

1. The next step is to divide the number in row 10 by the number in row 9. With any two starting numbers, you will find that the first three digits of the answer will always be 1.61.
2. In fact, if you were to continue this process to 20 lines or more and divide the 20th number by the 19th number, you would find ratios getting closer and closer to 1.618…, the golden ratio.
3. In this trick, we're dealing with two unknown quantities, so let's call the first two numbers chosen x and y. Now, our sequence of additions looks like the table on the right above.
4. The coefficients in each equation are Fibonacci numbers.
5. The sum of lines 1 through 10 is $55x + 88y = 11(5x + 8y)$.
6. Interestingly, the equation $5x + 8y$ is the same as the equation in line 7. To find the sum of lines 1–10, then, I simply multiplied the result in line 7 by 11: $11 \times 61 = 671$.
   a. You can easily multiply any two-digit number by 11 as follows: Using 11×61 as an example, add 6 + 1 and insert the answer, 7, between the 6 and the 1: 671 is the answer. What happens if the numbers add up to something greater than 9? Try 11×85: 8 + 5 = 13; we insert the 3 in the middle, then carry the 1 to the 8 to get the answer 935.
   b. Why this method works is easy to see if we look at how we would normally multiply 61×11 on paper:

$$\begin{array}{r} 61 \\ \times\ \underline{11} \\ 61 \\ +\ \underline{610} \\ 671 \end{array}$$

   We see that $1 \times 61 = 61$ and $10 \times 61 = 610$; when we add these two results, we get a 6 on the left and a 1 on the right and, in the middle, 6 + 1.
7. Returning to the magic trick, we saw how to obtain the sum of lines 1 through 10, but how did we get 1.61 when we divided line 10 by line 9? The answer is based on adding fractions badly.
   a. If you didn't know how to add fractions correctly, you might add the numerators together and the denominators together; thus, $\frac{1}{3} + \frac{2}{5} = \frac{3}{8}$. Of course, this answer isn't correct, but it is true that when you add fractions in this way, the answer you get will lie somewhere in between the two original fractions.
   b. In general, if we add the numerators and the denominators for $a/b < c/d$, then the resulting fraction (called the *mediant* of those two numbers), $\frac{a+c}{b+d}$, will lie in between.

8. The number in line 10 of our magic trick is $21x + 34y$. The number in line 9 is $13x + 21y$. We're interested in that fraction: $\frac{21x+34y}{13x+21y}$. This is the mediant, the "bad fraction" sum, of $\frac{21x}{13x}+\frac{34y}{21y}$.
   a. In the fraction $\frac{21x}{13x}$, the x's cancel, leaving us with $\frac{21}{13}$ on the left, which is 1.615…
   b. On the right, we have $\frac{34y}{21y}$, which reduces to $\frac{34}{21}$, or 1.619…
   c. As long as the numerator and denominator are positive, then the mediant is guaranteed to lie in between 1.615 and 1.619. As you recall, I asked only for the first three digits of the answer, which is how I knew it was 1.61.

II. Algebra was the invention of an Arab mathematician named Al-Khowarizmi [also written al-Khwarizmi; c. 780–c. 850] around 825.

A. Al-Khowarizmi wrote a book, *Hisâb al-jabr w'al muqâbalah*, literally meaning the science of reunion and the opposition. Later on, it was interpreted as the science of transposition and cancellation. *Al-jabr* is where we get the term *algebra*, and *algorithm*, which is any formal procedure for calculating, was also coined in honor of Al-Khowarizmi.

B. Let's turn now to one of those word problems we all dreaded in school: Find a number such that adding 5 to it has the same effect as tripling it.
   1. We don't know the number yet, so let's call it x. If we triple x, we get $3x$, and we want $3x$ to be the same as $x + 5$, or $3x = x + 5$.
   2. First, we want to clean up this equation, but we have to keep in mind the golden rule of algebra: Do unto one side as you would do unto the other. Thus, we have to subtract x from both sides: $3x - x = x + 5 - x$. The left side is $3x - x = 2x$, and the right side is $x + 5 - x = 5$. Now we have a much simpler equation: $2x = 5$.
   3. We now need to divide both sides by 2. This leaves us with x on the left side and 5/2, or 2.5, on the right.
   4. Let's verify the answer: If we triple 2.5, we get 7.5, and if we add 5 to 2.5, we also get 7.5.

C. Here's another word problem: Find a number such that doubling it, adding 10, then tripling it will yield 90.
   1. Again, let's call the original number x. The first step is to double this number and add 10: $2x + 10$. Then, we have to triple that quantity, and it should equal 90: $3(2x + 10) = 90$.
   2. We simplify that equation by dividing both sides by 3. On the left, we then have $2x + 10$. On the right, we have 30.
   3. Next, we subtract 10 from both sides; the result is $2x = 20$. Of course, now we divide by 2 and we're left with $x = 10$.

D. Here's another word problem: Today, my daughter Laurel is twice as old as my daughter Ariel. Two years ago, Laurel was three times as old as Ariel. The question is: How old are they today?
   1. Here, we have two unknowns, Laurel's age today and Ariel's age today. We'll call those unknowns L and A, respectively.
   2. We know that today Laurel is twice as old as Ariel: $L = 2A$. We also know that two years ago, $L - 2$, Laurel was three times as old as Ariel, $A - 2$. This sentence translates into the equation $L-2=3(A-2)$. The right side of the equation, $3(A - 2) = 3A - 6$; thus, $L - 2 = 3A - 6$.
   3. Let's now substitute what we learned from the first equation: $L = 2A$. Wherever we see L, we can replace that term with $2A$. The left side of the second equation, then, reads $2A - 2$; the right side still reads $3A - 6$: $2A - 2 = 3A - 6$.
   4. Now, we can simplify by adding 6 to both sides to eliminate the 6 on the right: $2A - 2 + 6 = 3A - 6 + 6$, or $2A + 4 = 3A$.
   5. Subtracting $2A$ from both sides leaves us with $4 = A$. Therefore, Ariel is 4, and Laurel, who is twice Ariel's age today, is 8.

III. The last technique we'll learn in this lecture is FOIL, which we use when we're multiplying several variables together.

A. Suppose we want to multiply the quantity $(a + b)$ by the quantity $(c + d)$. We can write the equation with the answer as follows: $(a + b)(c + d) = ac + ad + bc + bd$.

B. When we multiply the first numbers in the two sets of parentheses, we get ac. When we multiply the outer numbers in the two sets of parentheses, we get ad. We get bc when we multiply the inner numbers and bd when we multiply the last numbers. The name **FOIL** comes from this technique of multiplying **first, outer, inner, last**.

C. FOIL is nothing more than the *distributive law*. According to this law, we can look at $(a+ b)(c + d)$ as $a(c + d) + b(c + d)$. By the distributive law again, we can look at $a(c + d)$ as $ac + ad$ and $b(c + d)$ as $bc + bd$. If we put that all together, we get $ac + ad + bc + bd$, which is FOIL.

D. Let's do an example to solidify that concept: 13×22. The way we would do multiplication on paper is really nothing more than an application of FOIL, the distributive law. That is, 13 is $(10 + 3)$ and 22 is $(20 + 2)$. Multiplying these two quantities together, we get 10×20 for the first term, 10×2 for the outer, 3×20 for the inner, and 3×2 for the last. The result is $200 + 20 + 60 + 6 = 286$.

E. Let's do a few other examples.
   1. First, let's look at $(x + 3)(x + 4)$. The FOIL results for this example would be: x^2, $4x$, $3x$, and 12. Adding those together, we get $x^2 + 4x + 3x + 12$, or $x^2 + 7x + 12$.
   2. Now, let's look at $(x + 6)(x - 1)$. Think of $x - 1$ as $x + (-1)$. The FOIL results for this example would be: x^2, $-x$, $6x$, and -6. Adding those together (*combining like terms*) gives us $x^2 + 5x - 6$.
   3. Finally, let's look at $(x + 3)(x - 3)$. The FOIL results would be x^2, $-3x$, $+3x$, and -9. The $-3x$ and $+3x$ cancel, leaving us with $x^2 - 9$.

F. You can see, going through the same kinds of calculation, that if we multiply $(x + y)(x - y)$, we get the expression $x^2 - y^2$. We'll see applications of that equation in our next lecture.

Reading:

I. M. Gelfand and A. Shen, *Algebra.*

Peter H. Selby and Steve Slavin, *Practical Algebra: A Self-Teaching Guide*, 2nd ed.

Rich Barnett and Philip Schmidt, *Schaum's Outline of Elementary Algebra*, 3rd ed.

Questions to Consider:

1. Pick any two different one-digit numbers and a decimal point. These numbers can be arranged in six different ways. (For instance, if you choose the numbers 2 and 5, you can obtain 52, 25, 5.2, 2.5, .52, and .25.) Next, perform the following steps: Add the six numbers together, multiply that sum by 100, divide by 11, divide by 3, and finally, divide by the sum of the original two numbers you selected. Your answer should be 37. Why?

2. Choose any three-digit number in which the numbers are in decreasing order (such as 852 or 931). Reverse the numbers, and subtract the smaller number from the larger. (Example: $852 - 258 = 594$.) Now, reverse the new number you just got and add the two numbers together. (Example: $594 + 495 = 1{,}089$.) Use algebra to show that your final answer will always be 1089.

Lecture Seven

The Joy of Higher Algebra

Scope: We begin this lecture by showing how an algebraic formula allows us to square and multiply numbers in our heads very quickly. In the previous lecture, we learned how to solve for an unknown variable from a linear (first-degree) equation; here, we learn to solve second-degree equations using the technique of completing the square and the quadratic formula. We apply the quadratic formula to understand the connection between Fibonacci numbers and the golden ratio. We mention the history of the search for formulas for the solutions to third-degree and fourth-degree equations, culminating with Abel's proof that it is impossible to find a simple formula for equations of the fifth degree or higher.

Outline

I. Let's begin by looking at the equation we saw at the end of Lecture Six, namely, $(x + y)(x - y) = x^2 - y^2$. This equation can help you learn to square numbers in your head faster than you ever thought possible.

   A. In Lecture Two, we learned how to square numbers that end in 5. For example, to square 65, we know that the answer will end in 25 and begin with the product of 6 and 7, which is 42. The answer is 4,225.

   B. Let's start off with an easy number, 13. The number 10 is close to 13, and it's easier to multiply. We'll substitute 10, then, for 13, but we have to keep in mind that if we go down 3 to 10, we must go the same distance up, which gives us the number 16. Instead of multiplying 13×13, we'll multiply 10×16. The result is, of course, 160, and to that, we add the square of the number 3, the distance we went up and down; 3^2 is 9, and $160 + 9 = 169$, or 13^2.

   C. If we do a problem that ends in 5, such as 35^2, we can see why the answer turns out to be the same as it did with our earlier trick. The nearest easy number could be 30 or 40; we go down 5 to 30 and up 5 to 40: $30 \times 40 = 1{,}200$. Now we add 5^2, which is 25, to get 1,225.

   D. Let's try one final example, 99^2. We'll go up 1 to 100 and down 1 to 98: $98 \times 100 = 9{,}800$. To that, we add 1^2, or 1, which means that the answer is 9,801.

   E. The reason this trick works is all based on algebra. Let's start with the equation $x^2 = x^2 - y^2 + y^2$.
      1. The $-y^2$ and $+y^2$ cancel, leaving us with x^2. As we saw at the end of the last lecture, $x^2 - y^2$ is equal to $(x + y)(x - y)$; that means, then, that x^2 is equal to $(x + y)(x - y) + y^2$.

2. For clarity, let's substitute in the number we just used in the last example, 99^2. If we let $y = 1$, then we have $99^2 = (99+1)(99-1) + 1^2$; that simplifies to $(100\times 98) + 1$, which gives us 9,801.

II. Let's turn to a trick that is even more magical, and it's based on similar algebra. This works for multiplying two numbers that are close together.

A. We'll start with 106×109. The first number, 106, is 6 away from 100; the second number, 109, is 9 away from 100. Now, we add 106 + 9 or 109 + 6, which is 115. Next, we multiply 115 by our easy number, 100: $115\times 100 = 11{,}500$. Then, multiply 6×9 and add that result to 11,500 for a total of 11,554.

B. Again, this trick works through algebra.

1. Let's suppose we're multiplying two numbers, $(z + a)$ and $(z + b)$. Think of z as a number that has lots of zeros in it. If we multiply the numbers using FOIL, we get: $(z + a)(z + b) =$ $z^2 + za + zb + ab$.

2. Notice that the first three terms have z's in them, so we can factor out a z to get $z(z + a + b) + ab$.

3. Let's substitute numbers, say, 107×111: $(100 + 7)(100 + 111)$. When z is 100, the answer will be $100(100 + 7 + 111) + (7\times 11)$. To solve that, we first get 11,800; then we add 7×11 for a total of 11,877.

C. Let's do another example: 94×91. These numbers are both close to 100, but they're less than 100. The number 94 is –6 away from 100, and the number 91 is –9 away from 100. We now subtract 94 – 9 or 91– 6 to get 85. We multiply 85×100 to get 8,500. To that answer we add $(-6)(-9)$, which is +54. Our answer, then, is 8,500 + 54 = 8,554.

D. What if one of the numbers is above 100 and one of them is below 100? Let's try 97×106. The number 97 is 3 below 100, and 106 is 6 above 100. We start by adding 97 + 6 = 103. We then multiply $103\times 100 = 10{,}300$. To that number, we add $(-3)(6)$, which is to say that we subtract 18 from 10,300: 10,300 – 18 = 10,282.

E. Let's try a simpler problem: 14×17. The nearest easy number to 14 and 17 is 10. Here we multiply 10×21, which is 210. Now we add 4×7, because we were 4 away from 10 and 7 away from 10: 210 + 28 = 238.

F. Let's do one more of these problems: 23×28. Those numbers add up to 51, so we multiply $20\times 31 = 620$. To that, we add 3×8, which gives us an answer of 644.

1. This trick is especially magical when the two numbers at the end add up to 10 because then the multiplication becomes so easy you almost don't have to keep track of the zeros.

2. With 62×68, we multiply 60×70 to get 4,200, to which we add 2×8 to get an answer of 4,216.

3. We're using the same trick that we did for squaring numbers that end in 5. Try 65^2. We're 5 away from 60, and 65 + 5 = 70. We multiply 60×70, which is 4,200, and add 5^2 to get 4,225.

III. Now we'll move on to solving quadratic equations.

A. Before we work on quadratic equations, let's have a quick refresher on solving linear equations.

1. For the equation $9x - 7 = 47$, we first add 7 to both sides, which leaves us with $9x = 54$. Dividing both sides by 9 gives us $x = 6$.

2. Now let's try $5x + 11 = 2x + 18$. Subtract 11 from both sides and subtract $2x$ from both sides, leaving $3x = 7$. Solving, we get $x = 7/3$.

3. We should also verify these solutions. In the first equation, we plug in 6 for x, then check to make sure that $9x - 7 = 47$. In the second equation, we plug in 7/3 for x: $5x + 11$ would be 35/3 + 11, which is 68/3; $2x + 18$ would be 14/3 + 18, and 14/3 + 54/3 is also 68/3.

B. Let's now solve a quadratic equation: $x^2 + 6x + 8 = 0$, using a technique called *completing the square*. Look at $x^2 + 6x + 8$; notice that $(x + 3)(x + 3) = x^2 + 6x + 9$, which would be a perfect square. We can turn our equation into that perfect square by adding 1 to both sides.

1. When we add 1 to both sides, we get $x^2 + 6x + 9 = 1$. We see that $x^2 + 6x + 9$ is the quantity $(x + 3)^2$, which means that quantity is 1.

2. There are only two numbers that yield 1 when squared: 1 or –1. Thus, it must be the case that $x + 3 = 1$ or -1. If $x + 3 = 1$, that means that $x = -2$. If $x + 3 = -1$, that means that $x = -4$.

3. If we plug in $x = -2$, we get $-2^2 + 6(-2) + 8 = 0$, which is true. If we plug in $x = -4$, we get $(-4)^2 + 6(-4) + 8 = 0$, which is also true.

C. Essentially, using the same logic, we can derive the *quadratic formula*. According to the quadratic formula, any equation of the form $ax^2 + bx + c = 0$ has the following solutions: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

1. Let's look at this in terms of the last equation we did: $x^2 + 6x + 8 = 0$. The coefficient behind the x^2, that's a, is equal to 1; the coefficient behind the x, that's b, is equal to 6; and the constant, term c, is equal to 8.

2. Plugging that into the quadratic formula, we get:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(+8)}}{2(1)} = \frac{-6 \pm \sqrt{36-32}}{2} = \frac{-6 \pm \sqrt{4}}{2}, \text{ or}$$

$x = \dfrac{-6 \pm 2}{2}$, meaning that $x = -4$ or -2.

D. Here's one more example: $3x^2 + 4x - 5 = 0$. Here, $a = 3$, $b = 4$, and $c = -5$. Plugging into the quadratic formula, we get:

$$x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{2(3)} = \frac{-4 \pm \sqrt{76}}{6} = \frac{-4 \pm 2\sqrt{19}}{6} = \frac{-2 \pm \sqrt{19}}{3}$$

E. Let's try to plug in the equation $x^2 + 1 = 0$. According to this equation, we're squaring a number and adding 1 to it, and the result is 0; that's impossible. Plugging this into the quadratic formula, a is 1, b is 0, and c is 1, and we see the result shown at right. Of course, $\sqrt{-4}$ has no solutions that are real numbers.

$$x = \frac{0 \pm \sqrt{0 - 4(1)(1)}}{2(1)} = \frac{0 \pm \sqrt{-4}}{2}$$

F. Let's look at another application of quadratics, called *continued fractions.*

1. Look at the fraction $1+\frac{1}{1}$. That fraction is equal to 2, which we can write as $\frac{2}{1}$. Now let's add $1+\cfrac{1}{1+\frac{1}{1}}$; the answer is $1+\frac{1}{2}=\frac{3}{2}$.
2. If we repeat the process, the answer is equal to 1 plus the reciprocal of $\frac{3}{2}$, $\frac{2}{3}$: $1+\frac{1}{3/2}=1+\frac{2}{3}=\frac{5}{3}$. Where are we going?
3. By looking at these series of 1s, we're getting the fractions $\frac{2}{1}$, $\frac{3}{2}$, $\frac{5}{3}$; let's do the addition one more time: $1+\frac{1}{5/3} = 1 + \frac{3}{5} = \frac{8}{5}$.
4. Of course, we see the Fibonacci numbers in the resulting fractions, and I claim that this pattern will continue to produce Fibonacci fractions.
5. Imagine that we have 1 + 1 over some other messy term. If that messy term reduces to, say, $1+\frac{1}{Fn/Fn-1}$, that simplifies to $\frac{F_n + F_{n-1}}{F_n}$. By definition, $F_n + F_{n-1}$ is equal to F_{n+1}. In other words, once we have the ratio of Fibonacci numbers and we repeat this process, we can't help but get a new ratio of Fibonacci numbers.

G. What does this have to do with quadratics? Suppose we were to continue this process forever. Let's call the result x and solve for x:.

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1...}}} = x$$

1. Notice that everything under the topmost fraction bar is itself equal to x; therefore x is equal to $1+\frac{1}{x}$.
2. To solve $x = 1+\frac{1}{x}$, multiply both sides by x for a result of $x^2 = x + 1$.
3. We subtract x and subtract 1 from both sides, leaving the equation $x^2 - x - 1 = 0$.
4. For the quadratic equation, $a = 1$, $b = -1$, and $c = -1$. Solving, we get two solutions (in box):

$$x = \frac{1+\sqrt{5}}{2} = 1.618\ldots \text{ (golden ratio)}$$
$$x = \frac{1-\sqrt{5}}{2} = -0.618\ldots$$

5. Only one of these solutions will work, and that is the positive one. We know that the negative solution is incorrect, because we can't add a group of positive numbers and end up with a negative.
6. Incidentally, we have now proved that the ratio of Fibonacci numbers in the long run gets closer and closer to the golden ratio.

IV. This method of solving quadratic equations was known even by the ancient Greeks, but the ancient Greeks did not know how to solve equations in a higher degree, such as a cubic equation: $ax^3 + bx^2 + cx + d = 0$.

A. This problem was first solved by Girolamo Cardano (1501–1576), a mathematician and gambler with a rather shady past. Through various means, he discovered a formula for solving the cubic.

B. The search went on for a formula to solve any quartic equation, that is, an equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$. The formula for this was determined by an Italian mathematician named Lodovico Ferrari (1522–1565).

C. Then, the search went on for hundreds of years to find a solution to the quintic equation, an equation of fifth degree. Finally, in 1824, a Norwegian mathematician, Niels Abel (1802–1829), showed that the attempt to find a solution to the quintic equation was futile. It is impossible to find a single formula that uses nothing more than adding and multiplying and taking roots of coefficients to solve a quintic equation.

Reading:

I. M. Gelfand and A. Shen, *Algebra.*

Peter H. Selby and Steve Slavin, *Practical Algebra: A Self-Teaching Guide*, 2nd ed.

Rich Barnett and Philip Schmidt, *Schaum's Outline of Elementary Algebra*, 3rd ed.

Questions to Consider:

1. A little knowledge can sometimes be a dangerous thing. Find the flaw in the following "proof" that 1 = 2:

   Start with the equation $x = y$.
   Multiply both sides by x: $x^2 = xy$.
   Subtract y^2 from both sides: $x^2 - y^2 = xy - y^2$.
   Factor both sides: $(x + y)(x - y) = y(x - y)$.
   Divide both sides by $x - y$: $x + y = y$.
   Substitute $x = y$: $2y = y$.
   Divide both sides by y: $2 = 1$.
   Voila!!?!

2. Using the close-together method, mentally multiply each of the following pairs: 105×103, 98×93, and 998×993. Would you have found these problems easy without knowing this method?

Lecture Eight

The Joy of Algebra Made Visual

Scope: In this last lecture on algebra, we'll see what was once an earth-shattering idea, the idea of connecting algebra with geometry—how you could actually visualize an equation. We'll begin by looking at polynomials and the *law of exponents*, according to which $x^a x^b = x^{a+b}$. From there, we'll turn to graphing linear equations, and later in the lecture, we'll look briefly at parabolic and cubic equations. We'll see how we can use algebra to solve a problem in geometry or answer the everyday question of which long-distance phone plan you should choose. Finally, we'll learn the fundamental theorem of algebra and explore negative and fractional exponents and their graphs.

Outline

I. Before we look at the connection between algebra and geometry, let's talk about *polynomials*. Here are some examples: $8x^3 - 5x^2 + 4x + 7$, $x^{10} + 9x^2 - 3.2$, and so on; x, $x + 7$, even 7 by itself are polynomials.

   A. The *degree* of a polynomial is the largest exponent in the polynomial. For instance, the first polynomial above has degree 3 because of the x^3 term. The second polynomial has degree 10 because of the x^{10} term; x by itself has degree 1, as does $x + 7$. The constant polynomial has degree 0. We can think of that as $7(x^0)$. When dealing with polynomials, all the exponents must be whole numbers that are at least 0; no negative or fractional exponents are allowed.

   B. First, let's review the law of exponents: $x^a x^b = x^{a+b}$. Why is that true? Initially, we might think $x^a x^b$ should be x^{ab}. Let's do an example.

      1. According to the law of exponents, $x^2 x^3 = x^{2+3} = x^5$. If we look at x^2, that's xx, and x^3 is xxx; when we multiply them together, we get $(xx)(xxx)$. That's 5 x's.

      2. We also want the law of exponents to be true when the exponent is 0. That is, $x^a x^0$ should equal x^{a+0}, but $a + 0$ is a, which means that $x^a x^0 = x^a$. If we want the law of exponents to work in this case, then x^0 must be 1 so that $x^a x^0$ is still x^a. That's the reason that $x^0 = 1$.

   C. A typical polynomial of degree n looks like this: $ax^n + bx^{n-1} + cx^{n-2} + \ldots$, in which the a, b, c, and so on—all the *coefficients*—can be any real numbers, integers or fractions. The only requirement is that if the equation is of degree n, the coefficient behind the x^n cannot be 0; if it were 0, the equation wouldn't have degree n but a smaller degree.

II. Now let's explore how we can actually see an equation. We'll start by looking at first-degree equations, that is, linear equations.

A. Let's take one of the simplest linear equations: $y = 2x$, and plug in some values. When x is 0, y is twice 0, which is 0. When x is 1, y is twice 1, which is 2. When x is 2, y is 4. When x is 3, y is 6. Notice that every time we add 1 to x, we add 2 to y. We'll now plot these points on the *Cartesian plane* (discovered by René Descartes), in which the horizontal axis is the x-axis and the vertical axis is the y-axis.

1. For instance, when we plot (3,6), that means we go three to the right on the x-axis and six up on the y-axis; where those coordinates meet is the point (3,6).
2. If we plot all the points and connect the dots, the result is a line that goes through the points; that's why this is called a *linear equation.*

B. Let's now change this equation to $y = 2x+ 3$. What does that do to the graph? It adds 3 to the same points that we had earlier. The new line is parallel to the old line, but it's now higher than the old line by 3. This graph is called the *graph of the function* $y = 2x+ 3$. We also give names to the coefficients behind the x term and the constant term; in this equation, $y = 2x+ 3$, 2 is the *slope* of the line and 3 is the *y intercept*.

1. The slope tells us how much the line is increasing. As we said earlier, if x increases by 1, then y increases by 2. If x decreases by 1, then y decreases by 2.
2. The y intercept tells us where the line crosses the y-axis; in this case, that point is (0,3). The first coordinate will always be 0, and the second coordinate will be the y intercept.
3. Let's generalize this by looking at the equation $y = mx + b$. The slope of this equation is m and the y intercept is b. For the y intercept, that means that the line will cross the y-axis at the point where $x = 0$ and $y = b$. The fact that the slope is m tells us that if x increases by 2, y will increase by $m \times 2$, or $2m$.

C. Let's draw some of these graphs to get a picture of them. Suppose we look at the equation $y = .5x - 4$. This equation tells us that the slope is 1/2. This line intercepts the y-axis at –4. We plot the y intercept at (0,4), and every time we increase x by 1, y increases only by 1/2.

D. Let's look at another line: $y = -4x + 10$. This line intercepts the y-axis at 10. For every increase of x by 1, the function decreases at a rate of 4.

E. How about a line with 0 slope? Let's plug in a random constant, say $y = 1.618$. We then have a line with 0 slope. By the way, that's still called a linear equation, even though it's an equation of 0 degree.

F. Finally, let's look at a line of infinite slope. Suppose we have the equation $x = 2$. That says $x = 2$ no matter what y is; y could be 0, 1, 100, $-\pi$, and x will always be 2. Plotting the result gives us a vertical line.

III. Let's now solve a geometry problem using algebra.

A. We start with two equations: $y = 2x + 3$ and $y = -4x + 10$. Where those lines cross, y is equal to both $2x + 3$ and $-4x + 10$. Let's then set those two equations equal to each other, that is, $2x+3 = -4x+10$, because where they meet, those two quantities are equal.

1. Now we add $4x$ to both sides and subtract 3 from both sides, resulting in $6x = 7$. Solving that, we get $x = \frac{7}{6}$.
2. At the point where the lines cross, remember that y is equal to $2x + 3$ or $-4x + 10$. For $2x + 3$, $y = 2\left(\frac{7}{6}\right) + 3$, or $\frac{7}{3} + 3$, or $\frac{16}{3}$.
3. To verify the solution, when x is equal to $\frac{7}{6}$, where is it on the line at $-4x + 10$? Solving, $-4\left(\frac{7}{6}\right) = \frac{-28}{6} + 10$, or $\frac{32}{6}$, or $\frac{16}{3}$.

B. Here's a more practical question: Suppose you were offered two phone plans, and you want to decide which of those plans will save you more money in the long run. One of the plans charges a \$10.00 flat fee, plus \$0.15 for each minute you use. The other plan charges a \$20.00 flat fee, plus \$0.10 for every minute. Which plan should you choose?

1. If you use the phone a lot, then you may want to pay the \$20.00 flat fee and get a lower rate of \$0.10 per minute. If you use your phone only a little, then you may want the \$0.15-per-minute rate with a smaller flat fee. To find out where the critical point is, we set these two equations equal to each other.
2. The first bill, B, is equal to \$10.00 + \$0.15M, M being the number of minutes you use. The second bill is \$20.00 + \$0.10M. Setting those two equations equal to each other, we get \$10.00 + \$0.15M = \$20.00 + \$0.10M. Putting the M's on one side and the constant terms on the other, we get \$0.05M = \$10.00. We then multiply both sides by \$20.00 to get $M = 200$.
3. If you use 200 minutes or more, then you want the plan that has the lower per-minute rate. If you use under 200 minutes per month, then you want the plan that has the \$10.00 fee, plus \$0.15 a minute.
4. Again, the solution is worth verifying. In this case, if you used 200 minutes, whether you use the first plan or the second plan, your bill would be \$40.00, which corresponds to the point on the graph where those two lines cross, (200,40).

IV. Let's graduate from first-degree equations to second-degree equations.

A. The equation $y = x^2$ is called a *quadratic equation*, and it's the simplest of second-degree equations. The graph that's drawn from this equation looks like a parabola.

B. If we change the equation to $y = 2x^2$, the graph still has the same basic shape, but y increases much faster than it did in the first equation. If we change the equation to $y = x^2 + 2$, we increase y by 2 everywhere.

C. The equation $y = (x - 2)^2$ will shift the parabola two units to the right. To see why the graph moves to the right, we look at what happens when $x = 2$. When $x = 2$, then $y = 0^2$, which is 0; thus, we shift to the right at the point on the parabola where $x = 2$.

1. The equation $y = (x + 2)^2$ will shift the parabola to the left. Notice that if we start with the earlier equation, $y = (x - 2)^2$, and we subtract 2, that brings the whole parabola down. The equation becomes $x^2 - 4x + 4 - 2$, or $x^2 - 4x + 2$, which looks like a generic quadratic equation: $y = x^2 - 4x + 2$.
2. Even though the second equation looks different from the equations we've seen before, it's nothing more than a shifted parabola. In fact, the same is true for any quadratic equation.
3. For instance, look at $y = x^2 - 8x + 10$. Using the technique of completing the square, we can rewrite that as $(x^2 - 8x + 16) - 6$, replacing the 10 with $16 - 6$. The quantity in parentheses is equal to $(x - 4)^2$. The equation can be written as $y = (x - 4)^2 - 6$, and the graph is a parabola shifted to the right by 4 and lowered by 6.
4. No matter what the second-degree equation is, it will result in a parabola that intersects the x-axis once, twice, or zero times.

V. Let's now look at third-degree equations.

A. Looking at the graphs for $y = x^3$, $y = x^3 + 4x^2 + 4x$, and $y = x^3 - 7x + 6$, we see that these cubics cross the x-axis at most three times.

B. The general rule for this is called the *fundamental theorem of algebra*, proved by Gauss. According to this theorem, the graph of a polynomial of degree n will intersect the x-axis at most n times. As we saw, for the quadratic equations, $n = 2$, and for the cubic equations, $n = 3$.

C. Equivalently, if $P(x)$ is a polynomial of degree n, the number of solutions to the equation $P(x) = 0$ is at most n. For instance, for the equation $2x^{10} - 7x^4 + 5x + 9 = 0$, the fundamental theorem of algebra tells us that there are at most 10 solutions to that problem.

VI. So far, we've been dealing with polynomials, which have exponents that are non-negative integers. Let's now look at some other kinds of exponents.

A. For instance, what does the quantity x^{-1} mean? If we want the law of exponents to be a law for all numbers, we want $x^a x^b = x^{a+b}$. What happens if we plug $a = -1$ and $b = +1$ into the law of exponents?

1. Plugging those values in, we get $x^{-1}x$ (that is, $x^{-1}x^1) = x^{-1+1}$, which is x^0, or 1. In other words, x^{-1} when multiplied by x gives us 1.
2. This means that x^{-1} is the reciprocal of x—that is, $x^{-1} = \frac{1}{x}$ for any x not equal to 0. For instance, 3^{-1} is $\frac{1}{3}$ and -7^{-1} is $\frac{1}{-7}$, or $-\frac{1}{7}$. But 0^{-1} is forever undefined.

B. Let's look at 3^{-2}, which is 3^{-1-1}, or $\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$, or $\left(\frac{1}{3}\right)^2$, or $\frac{1}{9}$. In other words, x^{-1} is $\frac{1}{x^1}$; x^{-2} is $\frac{1}{x^2}$. By the same logic, x^{-n} is $\frac{1}{x^n}$. Looking at the graph of $y = \frac{1}{x}$, for example, we see that as x gets closer to 0 from the right, $\frac{1}{x}$ gets closer to infinity. On the left side, as x gets closer to 0, y gets closer to negative infinity.

C. Using the law of exponents, what should $9^{1/2}$ mean? We want it to be true that $x^a x^b = x^{a+b}$ even when a and b are fractions.

1. By the law of exponents, $(9^{1/2})(9^{1/2})$ is $9^{1/2\,+\,1/2}$; it's also equal to 9^1, which is 9. That tells us that $(9^{1/2})(9^{1/2}) = 9$. In other words, $9^{1/2} = 3$.
2. You might think that if $9^{1/2} = 3$, then also $9^{1/2} = -3$ because $(-3)(-3) = 9$. However, we want $9^{1/2}$ to be well defined; thus, mathematicians define $9^{1/2}$ to be equal to 3, not -3. In general, $x^{1/2}$ is equal to $+\sqrt{x}$. For instance, $\sqrt{9} = 3$, $\sqrt{16} = 4$, $\sqrt{1} = 1$, $\sqrt{0} = 0$.
3. Let's look at a graph of $y = \sqrt{x}$. Notice that we see only the graph to the right of $x = 0$ because to the left of $x = 0$ are the square roots of negative numbers, which we're not quite ready to handle yet.
4. What should $x^{1/3}$ mean? By the law of exponents, $x^a x^b x^c = x^{1/3+1/3+1/3}$, which is x^1, or x. Therefore, $x^{1/3}$ is the cube root of x, which is denoted by $\sqrt[3]{x}$. For instance, $\sqrt[3]{8} = 2$, $\sqrt[3]{27} = 3$, and $\sqrt[3]{2} = \sqrt[3]{2}$, which numerically, is about 1.259. We can also look at the graph of the cube root function.

VII. Before we close, let's look at a couple of other important graphs.

A. For instance, we see the equation of the unit circle. The circle is centered around the *origin*—that's the (0,0) point—with a radius of 1; the equation for this is $x^2 + y^2 = 1$, or $y^2 = 1 - x^2$. Technically, we might say that the top half of the circle is $y = \sqrt{1-x^2}$ and the bottom half of the circle is $y = -\sqrt{1-x^2}$, but it's cleaner to put the top half and the bottom half together to get the equation $x^2 + y^2 = 1$.

B. Let's look at a more general circle. Instead of intercepting the x-axis and y-axis one away from the origin, suppose we intercept them r away from the origin; the equation then is $x^2 + y^2 = r^2$.

C. Here's another example: $x^2 + y^2 = 10^2$, or 100, would be a circle of radius 10. If we shift that circle two units to the right, the equation would be $(x - 2)^2 + y^2 = 10^2$. If we then pushed it up by one unit, the equation would be $(x - 2)^2 + (y - 1)^2 = 10^2$.

VIII. In this lecture, we've seen polynomials and how to graph them. We've also talked about the fundamental theorem of algebra and about negative and fractional exponents. In the next lecture, we'll see what joy we can find in the number 9.

Reading:

I. M. Gelfand and A. Shen, *Algebra.*

Peter H. Selby and Steve Slavin, *Practical Algebra: A Self-Teaching Guide*, 2nd ed.

Rich Barnett and Philip Schmidt, *Schaum's Outline of Elementary Algebra*, 3rd ed.

Questions to Consider:

1. At the start of a baseball game, your favorite player has a batting average of .200. During the game, he has two hits and strikes out twice. At the beginning of the next game, you notice that his batting average is now .250. How many hits has he had this season?
2. Speaking of batting averages, suppose player A has a better batting average than player B for two consecutive seasons. Must it be the case that player A's combined batting average for both seasons is better than player B's? Surprisingly, the answer is no. Can you find some numbers that support this paradox?
3. Use the fundamental theorem of algebra to prove that if two quadratic polynomials agree for three different values of x, then they must be equal. In general, show that if two n^{th}-degree polynomials agree for $n + 1$ different values of x, then they must be the same polynomial.

Lecture Nine
The Joy of 9

Scope: We begin this lecture with a magic trick: Think of a number, triple it, add 6, then triple that result. Next, add the digits of your answer. If you still have a two-digit number, add the digits again. You should now be thinking of the magical number 9. The reason this trick works is based on algebra and the fact that the digits of any multiple of 9 must sum to a multiple of 9. This is one of the wonders of *modular arithmetic* (sometimes called *clock arithmetic*), in which numbers can be thought of as wrapping around in a circle. We apply modular arithmetic in this lecture to learn the technique of *casting out 9s*, which is a handy way of checking your answers to arithmetical problems. We also learn a method to mentally compute the day of the week of any date in history.

Outline

I. We begin by finding out why the magic trick worked.

A. Let's call the first number that you thought of x. Tripling x gives you $3x$. When you add 6, you get $3x + 6$. When you triple that result, you get $3(3x + 6)$, or $9x + 18$. That last equation, $9x + 18$, is the same as $9(x + 2)$; thus, the number you get is guaranteed to be a multiple of 9.

B. Let's see what the first several multiples of 9 have in common.

1. You may have learned in elementary school that if a number is a multiple of 9, its digits will sum to 9 or a multiple of 9. For example, adding the digits in 18 yields $1 + 8 = 9$, as does adding the digits in 27: $2 + 7 = 9$. The rule is: A number is divisible by 9 if and only if the sum of its digits is a multiple of 9.
2. Let's do an example: 3,456. Adding the digits together, we get 18, and 18 is a multiple of 9; therefore, 3,456 is a multiple of 9.
3. What about the number 1,234? Its digits add to 10, and if we add the digits in 10, we get 1; therefore, 1,234 is not a multiple of 9. However, that 1 is the remainder when we divide 1,234 by 9.
4. This same rule works, by the way, for multiples of 3. A number is divisible by 3 if and only if its digits add up to a multiple of 3.

C. Look again at 3,456; this number is $(3\times1{,}000)+(4\times100)+(5\times10)+6$.

1. We can break 1,000 into 999 + 1, we can break 100 into 99 + 1, and we can break 10 into 9 + 1; and 6 stays 6. If we expand on this, we get $(3 \times 999) + (4 \times 99) + (5 \times 9)$, and we're left with a dangling 3×1, which is 3; 4×1, which is 4; 5×1, which is 5; and 6.
2. We know that 3×999 is a multiple of 9, 4×99 is a multiple of 9, and 5×9 is a multiple of 9; thus, all those combine to be some

multiple of 9, plus we have $3 + 4 + 5 + 6$, which is 18, also a multiple of 9, and adding 18 to a multiple of 9 still gives us a multiple of 9.

3. With 1,234, the same idea applies. This number is $(1 \times 1{,}000) + (2 \times 100) + (3 \times 10) + 4$. Expanding the 1,000s, 100s, and 10s as we did before, we get $(1 \times 999) + (2 \times 99) + (3 \times 9)$ (all of which are multiples of 9), plus we have $1 + 2 + 3 + 4$, which equals 10, but 10 can be broken into $9 + 1$. That leaves us with (a multiple of 9) + 9 + 1; therefore, as promised, 1,234 is 1 greater than a multiple of 9.

D. We can use this idea about 9s to check addition, subtraction, and multiplication problems. Suppose we want to add 3,456 + 1,234. Let's check the answer, 4,690, using a process called *casting out 9s*.

1. We reduce the number 3,456 by adding all the digits together, giving us 18; we then reduce 18 by adding its digits together to get 9. Then, we reduce 1,234 by adding its digits to get 10, and we add the digits of 10 to get 1. We've changed the original problem, 3,456 + 1,234, to the easier problem of $9 + 1 = 10$, and we add the digits of that answer to get 1. When we check our original answer, 4,690, we should get a 1 at the end of the process.
2. The digits of 4,690 add up to 19; the digits of 19 up to 10; and the digits of 10 add up to 1. Because we got a match, we can have confidence in our answer. If the ending numbers did not match, we'd know that we had made a mistake. Note, however, that we could get a match and still have an incorrect answer.
3. Why does this work? We know, from our earlier calculation, that 3,456 is a multiple of 9; it's $9x + 0$. We also know that 1,234 is $9y + 1$; therefore, when we add $9x + (9y + 1)$, we get $9(x + y) + 1$, which means that, in the end, the answer will reduce to 1.

E. Let's do a bigger problem: 91,787 + 42,864. If we add those numbers together correctly, we get 134,651. To check the answer, we add the digits of 134,651 to get 20; we then add the digits of 20 to get 2.

1. Next, we add the digits of 91,787 to get 32, which simplifies to 5, and we add the digits of 42,864 to get 24, which simplifies to 6. Finally, $5 + 6 = 11$, and those digits, $1 + 1$, add up to 2. Because the two numbers match, we can have confidence in our answer.
2. Again, this method won't reveal all mistakes. If we accidentally mix up two digits—for example, ending with 561 instead of 651—the numbers will still match, but the error won't be caught.

F. This method also works for subtraction problems, such as 91,787 – 42,864 = 48,923. If we add the digits of 48,923, we get 26; adding those digits, we get 8.

1. Adding the digits of the first number, 91,787, gives us 32; 32 then reduces to 5. The second number, 42,864, simplifies to 24, which reduces to 6. Because this is a subtraction problem, we subtract $5 - 6$, which gives us -1.
2. Remember that -1 is simply the remainder we get when we divide our answer to the original subtraction problem by 9. We can always change that number by adding or subtracting multiples of 9; thus, we'll add a 9 to -1 to get 8. The two reduced numbers match again.

G. Surprisingly, this method also works for multiplication problems. Let's multiply the same two numbers: $91{,}787 \times 42{,}864 = 3{,}934{,}357{,}968$. We first add the digits of that 10-digit number to get 57; we then add the digits of 57 to get 12 and the digits of 12 to get 3.

1. As before, 91,787 reduces to 5 and 42,864 reduces to 6. Because this is a multiplication problem, we multiply 5×6, which gives us 30. Those digits add up to 3, and we have a match again.
2. Why does this work? Basically, this algebraic statement explains it: $(9x + 5)(9y + 6) = 9(9xy + 5x + 6y) + 30$. According to this, if we have a number of the form $9x + 5$ and we multiply that by a number of the form $9y + 6$, we get a number of the form: 9(something) + 30, which is 9(something + 3) + 3.

II. The ideas behind this method actually extend beyond the number 9.

A. If we want to say 42,864 = 6 + some multiple of 9, the notation we use is $42{,}864 = 6 \pmod 9$. To clarify, we say that $a = b \pmod 9$ if $a = (b + \text{some multiple of } 9)$. In other words, $a = b + 9k$, where k is some integer. In general, we say that for any integer m (not just the number 9), $a = b \pmod m$ if $a = b +$ some multiple of m. Another way to say that is $a = b + mk$, where k is any integer. Yet another way of saying it is that the number m divides the difference of $a - b$.

B. We can do what is called modular arithmetic for any integer m. For example, using the same logic we used to demonstrate casting out 9s, we can show that if $a = b \pmod m$ and $c = d \pmod m$, then $a + c$ equals $b + d \pmod m$. Translated, that says that if a and b differ by a multiple of m and c and d differ by a multiple of m, then $a + c$ and $b + d$ will differ by a multiple of m. Moreover, $ac = bd \pmod m$. If we multiply $(a = b)$ by $(a = b)$, we get $a^2 = b^2 \pmod m$. Multiply that by $(a = b)$, and we get $a^3 = b^3$, $a^4 = b^4$, and in general, $a^n = b^n \pmod m$.

C. This may sound a bit abstract, but in fact, you do modular arithmetic every day. For instance, if the clock reads 12:00 right now, then what time will it read in 17 hours? You might reason as follows: 17 hours is 12 hours + 5 hours; ignoring the 12, the clock will read 5:00. What time will it be in 29 hours or 41 hours? To get the answer, we just add more multiples of 12, and we can ignore 12 when we're looking at a clock. We're working, then, in mod 12.

D. Here's another example: What will the clock read in 1,202 hours from 9:00? To find 1,202(mod 12), we go around the clock 100 times + 2; 1,202 is 2(mod 12). In 1,202 hours, the clock will read 9 + 2, or 11:00.

E. By working in mod 7, we can use this same approach to find the day of the week of any date in history.

1. First, we figure out the day of the week of any date in the year 2007. For this, we need to memorize a year code, which for 2007, is 0. Then, we need to memorize a code for every day of the week:

Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
1	2	3	4	5	6	7 or 0

Saturday is 7 or 0 because we're doing this in mod 7.

2. Next, we memorize a code for every month of the year. It's easiest to remember this code if you look at the months in groups of three.

Jan.	1	Apr.	0	July	0	Oct.	1
Feb.	4	May	2	Aug.	3	Nov.	4
Mar.	4	June	5	Sept.	6	Dec.	6
mnemonic:	$=12^2$		$=5^2$		$=6^2$		$=12^2+2$

3. Let's figure out the day of the week of December 25, 2007. Start with the month code for December, 6, and add 25 for the date. Then, for 2007, we add the year code, 0: 6 + 25 + 0 = 31.
4. We could count the days and wrap around the calendar until we get to 31, but we don't have to because every seven days, the week repeats. Day 31 will be the same as 31 minus any multiple of 7. The biggest multiple of 7 below 31 is 28, and 31 – 28 = 3; day 3 in the code is Tuesday. Thus, Christmas in 2007 will be a Tuesday.
5. We know that Thanksgiving 2007 will be a Thursday in November, but what will the date be? The month code for November is 4. We'll call the date x, and the year code for 2007 is 0. Our equation, then, is $4 + x + 0 = 5$, because Thursday is day 5. What do we add to 4 to get 5(mod 7)? We must add 1 or something that differs from 1 by a multiple of 7; thus, x will be 1, 8, 15, 22, or 29. The holiday occurs on the fourth Thursday of the month, or the 22^{nd}.

F. Why does this work? Think about what happens to your birthday as you go from one year to the next—it bumps up by exactly one day. That's because there are usually 365 days in between your birthdays, and 364 is a multiple of 7 ($7 \times 52 = 364$). The exception is that in a leap year, there are 366 days between your birthdays, unless you were born in January or February and the year hasn't leaped yet.

G. If we put all that together, we can figure out the year codes. Remember that 2007 has a year code of 0, but 2008 is a leap year; it will have 366 days. Thus, the year code for 2008 should be 2, except in January or February, when we have to subtract 1. The year code for 2009 is 3; for 2010, 4; for 2011, 5; and for 2012, another leap year, 7. Of course, we can reduce 7(mod 7) to 0, which means that 2012 has a year code of 0.

H. Incidentally, the year 1900 has a year code of 0, and knowing that fact, we can derive the year codes for every subsequent date.

1. How could we figure out the year code for 1961, for example? The year 1961 is 61 years after 1900; thus, the calendar will shift 61 times, but it will also shift an extra time for each leap year. There were 15 leap years between 1900 and 1960. We take 1/4 of 61, which is 15, then add 61 + 15, and that's 76.
2. We could make 76 the year code for 1961, but it's much simpler to look at 76(mod 7). Subtract the biggest multiple of 7 less than 76, 70, and we get 76 – 70 = 6, the year code for 1961. Thus, for March 19, 1961, we compute 6 + 4 +19 = 29, then subtract 28 for an answer of 1. So March 19, 1961 was a Sunday.

I. Let's look at one more example: July 22, 1987. We start by finding the year code for 1987: We take 1/4 of 87; that's 21 with a remainder of 3. In this trick, we always ignore the remainder. We add 87 + 21 to get 108 and subtract the biggest multiple of 7, which is 105. Next, 108 – 105 = 3; that's the year code for 1987. To that, we add 0 for the month code of July and 22 for the date: 3 + 0 + 22 = 35. Subtract the biggest multiple of 7, 21, for an answer of 4. July 22, 1987, was a Wednesday.

J. Here's one last challenge: Pick any four-digit number in which the digits aren't all the same. Let's use 1,618. Now, scramble those numbers to get a different number, such as 8,611. Subtract the smaller number from the larger number. In this case, you'll get 6,993. Next, add the digits: 6 + 9 + 9 + 3 = 27. If you have a two-digit number, add the digits again to get a one-digit number. The resulting number is 9.

Reading:

Martin Gardner, *The Second Scientific American Book of Mathematical Puzzles and Diversions*, pp. 43–50.

Martin Gardner, *Mathematics, Magic, and Mystery.*

Benedict Gross and Joe Harris, *The Magic of Numbers*, chapters 15–16.

Benjamin Arthur and Michael Shermer. *Secrets of Mental Math*, chapters 6, 9.

Edward M. Reingold and Nachum Dershowitz. *Calendrical Calculations*.

Questions to Consider:

1. If you take any number and scramble its digits, then subtract the original number from the scrambled one, you always get a multiple of 9. Why?
2. In the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,..., explain why every fifth number must be divisible by 5.

Lecture Ten
The Joy of Proofs

Scope: One of the things that makes mathematics so special is that while other subjects rely on experimental evidence to reach conclusions, mathematics allows some facts to be established with absolute certainty using various proof techniques. We begin by proving intuitive results, such as an even number plus an even number is even and an odd number times an odd number is odd. We then use this to explain why $\sqrt{2}$ can never be expressed as a fraction of two integers. We also discuss the ideas of an *existence proof* and *proof by induction*, leading to the explanation of various number patterns. We end the lecture with choice checkerboard challenges.

Outline

I. Let's start with something that we already know to be true to give an example of an easy proof. We'll use the statement that an even number plus an even number always equals an even number.

A. We say that an integer, a, is an even number if $a = 2b$, where b is any integer. Here's our theorem: If x and y are even numbers, then $x + y$ is also an even number. The following is our proof.

1. Because x is an even number, then $x = 2a$, where a is an integer. Because y is an even number, then $y = 2b$, where b is an integer. Thus, $x + y = 2a + 2b$, or $2(a + b)$.
2. The quantity $a + b$ is an integer. We know that because a was an integer and b was an integer; therefore, their sum is an integer. Because $x + y$ is twice an integer, $x + y$ is even.
3. Completed proofs end with a filled or empty box sometimes known as a Halmos symbol (■); or with QED (*quod erat demonstrandum*; humorously translated as "quite easily done"), or with ☺.

B. For the next proof, note that an integer is an odd number if the number is not even. A more mathematical statement of that is: An odd number a is a number that's of the form $2b + 1$. Here's the theorem we'll prove: If x and y are odd numbers, then their product is an odd number.

1. If x is an odd number, then x is of the form $2a + 1$. If y is an odd number, then $y = 2b + 1$, where a and b are integers; therefore, their product, xy, is $(2a + 1)(2b + 1)$.
2. When we multiply those quantities, we get $4ab + 2a + 2b + 1$; notice that the first three terms are all divisible by 2. Simplifying, we get $2(2ab + a + b) + 1$. Thus, xy is twice an integer plus 1; therefore, xy is an odd number.

C. Let's graduate from numbers that are even and odd to rational numbers. Those are fractions that are obtained by taking the quotient of two integers. Specifically, we say that a number r is rational, if $r = \frac{p}{q}$, where p and q are integers; of course, q cannot be 0 because that would mean we were dividing by 0. Here's our theorem to prove: The average of two rational numbers is always rational.

1. We'll use r_1 as a rational number in the form $\frac{p_1}{q_1}$ and r_2 in the form $\frac{p_2}{q_2}$; their average is $\frac{r_1 + r_2}{2}$. We add the fractions, $\frac{p_1}{q_1} + \frac{p_2}{q_2}$, and divide the result by 2. The result is $\frac{p_1q_2 + p_2q_1}{2q_1q_2}$. The numerator is the product and sum of integers; the denominator is the product of integers.
2. Because we have an integer in the numerator and the denominator, we know that the average of those two numbers, r_1 and r_2, is rational. The consequence of that statement is that between any two rational numbers, we can find another rational number.

D. If you were to look at the number line, you might believe, as some of the ancient Greeks once did, that all numbers are rational, that is, that every number out there is equal to some fraction. Pythagoras himself believed that and would have been surprised to learn that one of the numbers derivable from his own theorem was not a rational number.

1. According to the Pythagorean theorem, for any right triangle with side lengths a and b and hypotenuse c, $a^2 + b^2 = c^2$.
2. Using the simplest of right triangles, with side lengths 1 and 1, the hypotenuse would have a length of $\sqrt{2}$ because $1^2 + 1^2 = 2$.
3. We can approximate $\sqrt{2}$ with a decimal expansion ($\sqrt{2} =$ 1.414213…), but we cannot write $\sqrt{2}$ exactly as a fraction.
4. We might think that $\sqrt{2}$ is not rational because its decimal expansion doesn't repeat, but how can you be sure that it doesn't repeat? That would also require proof.
5. To prove that $\sqrt{2}$ is irrational, we'll use a *proof by contradiction*. Suppose $\sqrt{2}$ were rational; if that were true, then it would be of the form $\frac{p}{q}$ in lowest terms because we can write all fractions in lowest terms.

6. The equation is $\sqrt{2} = \frac{p}{q}$. Squaring both sides, we get $2 = \frac{p^2}{q^2}$.
7. We can rewrite that equation as $p^2 = 2q^2$, but that says that p^2 is twice a number, which means that p^2 is even. If p^2 is even, then p must be even because if p were odd, its square would be odd. That means that the number p must be of the form $2b$ because it's an even number.
8. Let's return to our equation, $p^2 = 2q^2$, and replace p with $2b$. When we do that, we have the expression $(2b)^2 = 2q^2$. When we square $2b$, we get $4b^2$, and that's equal to $2q^2$.
9. Dividing both sides by 2, we get $q^2 = 2b^2$. Again, that means that q^2 is even, and if q^2 is even, then q is also even.
10. Now we have a problem. We tried to prove that $\sqrt{2}$ is a rational number, $\frac{p}{q}$ in lowest terms. Then, we showed that both p and q had to be even. The problem is that if p is even and q is even, then that fraction wasn't in lowest terms.
11. If we assume that $\sqrt{2}$ is rational and in lowest terms, we conclude that $\sqrt{2}$ is not in lowest terms; therefore, the only conclusion we can make is that $\sqrt{2}$ is not rational.

II. Next, we'll prove what some mathematicians call an *existence theorem*: There exist irrational numbers a and b such that a^b is rational. In other words, we can find an irrational number raised to an irrational power that yields a rational number. What makes this an existence proof is that we'll become convinced of its truth without knowing what a and b are.

A. We begin by asking a simple question: Is $\sqrt{2}$ raised to the power of $\sqrt{2}$ rational? If $\sqrt{2}^{\sqrt{2}}$ is rational, then both a and b would be irrational, yet a^b would be rational, and our proof would be complete.

B. What if, however, $\sqrt{2}^{\sqrt{2}}$ is irrational? In that case, we could let $a = \sqrt{2}^{\sqrt{2}}$ (which we're assuming is irrational), and we could let $b = \sqrt{2}$ (which we've just shown is irrational); thus, $a^b = \left(\sqrt{2}\right)^{\sqrt{2}}$.

C. According to the law of exponents, $(a^b)^c$ is the same as a^{bc}. When we apply that here, we have $\left(\sqrt{2}\right)^{\sqrt{2}(\sqrt{2})}$. But that's equal to $\sqrt{2}^2$, which is equal to 2. In this case, then, we found an a and a b such that $a^b = 2$, which is rational.

D. The first question we asked was: Is $\sqrt{2}^{\sqrt{2}}$ rational? If the answer was yes, then our proof was complete. If the answer was no, then by choosing a to be $\sqrt{2}^{\sqrt{2}}$ and b to be $\sqrt{2}$, our proof is also complete.

III. Now let's look at a proof technique called *proof by induction.*

A. We'll look at a problem that we saw in Lecture Two: What is the sum of the first n odd numbers? Recall that $1 = 1$, or 1^2; $1 + 3 = 4$, or 2^2; $1 + 3 + 5 = 9$, or 3^2; $1 + 3 + 5 + 9 = 25$, or 5^2; and so on. Will this pattern go on forever? The sixth odd number is 11. When we add that to 25, we get 36, which is 6^2. If we trust the first five results, then the sixth result will follow.

B. Suppose we notice that the sum of the first k odd numbers is k^2. In other words, since the kth odd number is $2k - 1$, we are asserting that $1 + 3 + 5 + \ldots + k^{\text{th}}$ (that's the number $2k - 1$) $= k^2$. Then, what will be the sum of the first $k + 1$ odd numbers? What is the next odd number? That's $2k + 1$. When we add that to the sum of the first k odd numbers, or k^2, we get $k^2 + 2k + 1$, which is $(k + 1)^2$. In other words, if the sum of the first k odd numbers is k^2, then it's unavoidable that the sum of the first $k + 1$ odd numbers will be $(k + 1)^2$.

C. Here's another example of a proof by induction:

1. Recall that the sum of the first n odd numbers, which we called triangular numbers, is equal to $\frac{n(n+1)}{2}$. If we're interested in summing the cubes of the first n numbers, we can find a pattern: $1^3 = 1$, $1^3 + 2^3 = 9$, and $1^3 + 2^3 + 3^3 = 36$. The results are all perfect squares of the triangular numbers; that is, the sum $1^3 + 2^3 + \ldots + 5^3 = 225$, or 15^2, which is equal to $(1 + 2 + 3 + 4 + 5)^2$, or $\left(\frac{5 \times 6}{2}\right)^2$.
2. We'll assert, then, that the sum of the cubes of the first n numbers equals the n^{th} triangular number squared; that is, $\frac{n^2(n+1)^2}{4}$.
3. We'll start with a base case. We see that the statement works for the number 1; undeniably, $1^3 = 1^2$. Then, we state our *induction hypothesis*: Suppose that the sum of the cubes of the first k numbers is $\frac{k^2(k+1)^2}{4}$. We'll use that fact to show that the statement will continue to be true when we look at the sum of the cubes of the first $k + 1$ numbers.

4. What do we want to see at the end of that? If we replace k with k + 1 in the above formula, we want to see $\dfrac{(k+1)^2(k+2)^2}{4}$.
5. The sum of the first $k+1$ cubes is equal to the sum of the first k cubes plus $(k+1)^3$. But what do we know about the sum of the first k^3? By our induction hypothesis, we know that quantity is equal to $\dfrac{k^2(k+1)^2}{4}$. We'll add to that the number $(k+1)^3$. When we add that, we can save ourselves a lot of messy algebra by factoring out the number $(k+1)^2$ because that divides both the first term and the second term. We're left with the results shown at right.

$$\boxed{\begin{gathered}(k+1)^2\left(\frac{k^2}{4}+\frac{4(k+1)}{4}\right)=\\(k+1)^2\left(\frac{k^2+4k+4}{4}\right)=\\\frac{(k+1)^2(k+2)^2}{4}\end{gathered}}$$

D. Let's try a question that doesn't involve any algebra or symbol manipulation: Can we cover an 8-by-8 checkerboard with non-overlapping L-shaped dominoes (called *trominoes*)?
1. An 8-by-8 checkerboard has 64 squares, and if a tromino takes up 3 squares, we won't be able to cover the board evenly, since 64 is not a multiple of 3.
2. If we remove any square at all from the checkerboard, can we cover the rest of the board with trominoes? In this case, 3 divides evenly into 63, so it might be possible. Let's prove that it is, in fact, possible, and that it's also true for 2-by-2 checkerboards, 4-by-4 and 8-by-8 checkerboards, and so on—any 2^n-by-2^n checkerboard.
3. Let's use a 2-by-2 board, or 2^1 by 2^1, as our base case. We can see that if we remove any square from this board, then the rest of the board can be covered with a single tromino.
4. We assume that this is true for any board of size 2^k by 2^k. We'll now see that it's true for any board of size 2^{k+1} by 2^{k+1}.
5. We have a checkerboard with dimensions 2^{k+1} by 2^{k+1}. We break that checkerboard into four quadrants, so that each quadrant is now of size 2^k by 2^k. Look at the quadrant that we deleted the square from. We know by the induction hypothesis that the rest of that quadrant can be covered with non-overlapping trominoes. But how do we cover those other three quadrants?
6. Let's look at one of the other three quadrants. We know that if we remove any square from that quadrant, then the rest of it can be covered with non-overlapping trominoes. Let's remove the square closest to the center of the board, then tile the rest of that quadrant. We can do the same for each of the remaining quadrants.
7. We've now covered the entire board except for three squares we removed near the center of the board. But those three squares form a tromino themselves. Thus, when we place a tromino over those squares, we've completely covered our 2^{k+1}-by-2^{k+1} checkerboard.
8. That proof is not only inductive but also constructive. It tells us how we can cover the 8-by-8 board. We could start with the 8-by-8 board, remove one square, and go through the procedure we outlined to systematically cover the rest of the board.

E. Could we tile an 8-by-8 checkerboard with dominoes? Dominoes have dimensions of 2 by 1; they cover two consecutive squares.
1. We could easily cover an 8-by-8 checkerboard with dominoes, but we couldn't do so if we removed a square from the checkerboard.
2. Could we cover the board with dominoes if we removed any two squares? We could cover the board if we removed two side-by-side squares, but what if we remove two squares in opposite corners?
3. A checkerboard has red squares and white squares; thus, any domino that we place on the board must cover a white square and a red square. When we removed two squares, we were left with 62 squares, which means that we would need 31 dominoes to cover the board. But the two squares that we removed were both the same color. The resulting board has 30 red squares and 32 white squares, so there's no way we could cover it with 31 dominoes.

IV. Let's end this lecture with a proof that all numbers are interesting.

A. For instance, 1 is the first number, which is obviously interesting. Then, 2 is the first even number, which makes it an interesting number, too. Because 3 is the first odd prime number, that's interesting. Then, 4 is the first and only number that spells itself: F-O-U-R.

B. Here's the proof that all numbers are interesting: Suppose that not all numbers were interesting; then, there would have to be a first number that wasn't interesting. But wouldn't that make that number interesting?

Reading:

Edward B. Burger, *Extending the Frontiers of Mathematics: Inquiries into Proof and Argumentation.*

Benedict Gross and Joe Harris, *The Magic of Numbers*, chapters 15–16.

Daniel J. Velleman, *How to Prove It.*

Questions to Consider:

1. Prove by induction that the sum of the first n Fibonacci numbers is one less than the $(n + 2)^{\text{th}}$ Fibonacci number. For example, $1 + 1 + 2 + 3 + 5 + 8 = 21 - 1$. Prove that the sum of Fibonacci numbers that are two apart is always a Lucas number (where the Lucas numbers are 2, 1, 3, 4, 7, 11, 18, 29,...).
2. Place a rook on any point on an 8-by-8 checkerboard. Show that it is possible to move the rook (making only horizontal or vertical moves) in such a way that it visits every square on the checkerboard exactly once and ends at the same point. Use this to prove that if we remove any two squares of opposite color from the checkerboard, then we can cover the remaining squares with 31 dominoes. (Hint: What can you say about the number of steps to walk from one square to another of the same color if only horizontal and vertical steps are allowed?)
3. Prove that the number log 2 is irrational, where the log is base 10. (Hint: Prove by contradiction.) In fact, except when n is a power of 10, log n is irrational.

Lecture Eleven

The Joy of Geometry

Scope: Geometry is one of the oldest subjects in mathematics. Applied by ancient Egyptians, its theory was formalized by the ancient Greeks. The first geometry textbook, written by Euclid, has inspired students for centuries. Geometry is based on a handful of definitions and axioms involving points, lines, and angles. From these elementary ideas, we derive important, nontrivial conclusions about the perimeters and areas of triangles and other polygonal shapes. In this lecture, we'll use these ideas to derive the Pythagorean theorem and determine the length of a piece of a rope that would stretch from the lower-left corner of the floor to the upper-right corner of the ceiling.

Outline

I. The term *geometry* is Greek and literally means "to measure the Earth." The oldest textbook on the subject is *The Elements* by Euclid. In this lecture, we'll learn how to measure lengths, angles, and areas.

   A. We begin by defining our terms and looking at basic geometric objects.

      1. A *point* is an infinitely small dot.
      2. A *line* is an infinite one-dimensional object and is named by naming two points that lay on the line ($\overleftrightarrow{AB}$).
      3. A *ray* is like a line except that it has one endpoint and proceeds out infinitely from that point in only one direction. It is named by giving the endpoint and one other point on the ray ($\overrightarrow{OA}$).
      4. A *line segment* is the portion of a line between two points ($\overline{AB}$).
      5. An *angle* results when two rays share the same endpoint. It is named by giving one point on one ray, the endpoint, and a point on the other ray ($\angle AOB$).
      6. Two lines are *parallel lines* ($\parallel$) if they never intersect.
      7. Two lines are *perpendicular lines* ($\perp$) if their intersection results in four equal, or right, angles.

   B. All of geometry is built from points and lines and comes from the following five axioms of Euclidean geometry:

      1. A straight line segment can be drawn joining any two points.
      2. Any line segment can be extended indefinitely.
      3. Given any line segment, a circle can be drawn with that segment as the radius and one endpoint as the center.
      4. All right angles are congruent and measure 90°.
      5. Given a line and a point not on the line, there is *exactly one* line through the point that is parallel to the original line. This is

equivalent to the axiom that the sum of the angles in any triangle is always equal to 180°.

C. Essentially, all of geometry can be built from those five axioms. For example, here's a theorem: Every straight line has an angle of 180°. Look at $\angle AOB$. We have a straight line that goes through A and B. We can bisect the line AB at a right angle by drawing a line from O to C; that line is called a *bisector*. We can now obtain $\angle AOB$ by adding $\angle AOC$ to $\angle COB$. Because both of those were right angles, then according to the fourth axiom, they both measure 90°. Adding 90° + 90° gives us 180°.

D. Let's prove the *vertical angle theorem*: Suppose AB and CD are lines that intersect at point O. Then $\angle \mathrm{AOC} = \angle \mathrm{BOD}$. (In other words, the measure of angle AOC equals the measure of angle BOD. Some authors write this as $\mathrm{m}\angle \mathrm{AOC} = \mathrm{m}\angle \mathrm{BOD}$.)

1. We know that the measurement of the line AB is 180°. That means that $\angle AOC + \angle COB$ must sum to 180°. Those two angles are called *supplementary angles* because they add up to 180°.
2. Similarly, if we look at the line COD, we see that $\angle COB + \angle BOD$, because they form a straight line, also sum to 180°.
3. We can now subtract these algebraic equalities to get: $\angle AOC - \angle BOD = 180 - 180 = 0$. In other words, $\angle AOC = \angle BOD$.

E. Let's look at the *corresponding angle theorem*: If L_1 and L_2 are parallel lines, and a third line crosses the pair (the *transverse line*), then the corresponding angles formed by the third line must be equal.

1. We'll prove that $\angle A$ and $\angle B$ are equal by drawing a new line from point C to create two new right triangles. By Euclid's fifth postulate, we know that the sum of the angles in any triangle is 180°. Thus, the angles in both of our triangles must sum to 180°.
2. Subtracting, we see that $\angle A - \angle B = 0$. That is, $\angle A = \angle B$.

II. We know that the sum of the angles of any triangle is 180°, but what about larger objects? Four-sided objects are called *quadrilaterals*, or *4-gons*. Five-sided objects or many-sided objects are called *polygons*.

A. The sum of the angles of any four-sided figure is 360°. By drawing a diagonal from one corner to the opposite corner of a four-sided figure, we create two triangles whose angles each sum to 180°; when we add the angles, we get 360°.

B. The sum of the angles in a pentagon is 540°. If we cut a small triangle off the top of the pentagon, we're left with a quadrilateral whose angles sum to 360°. When we add the angles of the extra triangle back in, we get 360° + 180° = 540°.

C. We're essentially doing a proof by induction here to show that for any n-sided polygon, the sum of the angles will be always be $(n-2)180°$.

III. *Perimeter* and *area* are two other terms used frequently in geometry.

A. The perimeter of an object is the sum of the lengths of its sides. For a rectangle with a base of length b and a height of length h, this is $2b+2h$.

B. We define the area of a 1-by-1 square to be 1. We then attempt to define all areas in terms of that unit quantity.

1. For instance, if we have a rectangle that has a height of 3 and a base length of 4, we can show that the area of that rectangle is 12 simply by cutting it into 12 squares of dimensions 1 by 1. Those squares all have area 1; therefore, the area of the rectangle is 12.
2. The area of a rectangle whose sides have positive lengths is $b \times h$.

C. We can also show that the area of any triangle with a base of length b and a height of length h has an area equal to $\frac{1}{2}(b \times h)$.

1. We adjoin two right triangles, each with a base of b and a height of h, to create a rectangle. The area of that rectangle is bh.
2. Given that the first triangle and the second triangle have the same area, then the area of the first triangle must be $\frac{1}{2}bh$.

D. It seems odd that any triangle will have an area of $\frac{1}{2}bh$. Imagine we have parallel lines and we place two points on the first line at a distance of b, for base. If those two parallel lines are separated by a distance of h, then no matter where we put the third point on the second line to create a triangle, the area will always be the same: $\frac{1}{2}bh$.

1. Let's look at a triangle with a base of length b and a height of length h. Let's now break that triangle up into two smaller triangles. The triangle on the left will have area a_1 and the triangle on the right will have area a_2.
2. The triangle on the left is a right triangle, and we know that its area is $\frac{1}{2}bh$. If we split the base into two parts, one part of length b_1 and the other of length b_2, then the area of the triangle on the left is $\frac{1}{2}b_1(h)$ and the area of the triangle on the right is $\frac{1}{2}b_2(h)$.
3. Therefore, the total area is $\frac{1}{2}b_1(h) + \frac{1}{2}b_2(h)$, which is equal to $\frac{1}{2}h(b_1 + b_2)$, but $b_1 + b_2$ was equal to b, the length of the original base. Thus, the total area of that triangle is $\frac{1}{2}bh$.

E. For a triangle with an obtuse base angle, we have a different proof.

1. Rather than breaking this triangle into two, we'll extend the line that has length $b_1 + b_2$ to create a larger right triangle. We know that the area of a right triangle is $\frac{1}{2}bh$.
2. The length of the base for this triangle is $(b_1 + b_2)$; the area of this triangle, then, is $\frac{1}{2}(b_1 + b_2)h$. The area of the new triangle we created, denoted a_2, is $\frac{1}{2}b_2(h)$ because it's a right triangle. The area of the original triangle, a_1, is what we get when we subtract the triangle with area a_2 from the larger triangle. That's equal to $\frac{1}{2}(b_1 + b_2)h - \frac{1}{2}(b_2)h$. Algebraically, the quantities $\frac{1}{2}(b_2)h$ cancel, and we're left with $\frac{1}{2}(b_1)h$.

IV. The Pythagorean theorem states that given a right triangle with side lengths a and b and hypotenuse length c, $a^2 + b^2 = c^2$. Note that the *hypotenuse* is the length of the side that is opposite the right angle.

A. Imagine that we adjoined four right triangles, each with side lengths a and b and hypotenuse c, to form a square with side lengths $a + b$.

1. In the middle of that square is another square whose sides all have length c.
2. One way we know that we have a square in the middle is by the symmetry of the object we've created: We see that all the angles of this four-sided figure must be equal and, thus, must sum to 360°. All of those angles, then, must be 90°, or right angles.

B. Let's start again with a picture of a big square with a little square in the middle. The area of the larger square is $(a + b)^2$.

1. We can also compute the area of the big square by finding the areas of the triangles plus the area of the square in the middle. The area of each of the right triangles is $1/2(ab)$; together, those areas add up to $4 \times 1/2(ab)$, plus the area of the square in the middle, or c^2.
2. When we set these two quantities equal to each other, we get: $a^2 + 2ab + b^2 = 2ab + c^2$. The $2ab$'s cancel, leaving $a^2 + b^2 = c^2$.

C. We can use the same theorem to find the length of any line segment.

1. Suppose we want to determine the length of a line segment between point (0,0) and point (4,3). If we draw a right triangle that starts at (0,0) and ends at (4,3), we know that the length of that line segment will satisfy the Pythagorean theorem. That is, $4^2 + 3^2$ will be the length of that line segment squared. Because that length squared must be 25, then the length of the line must be 5.
2. In general, we can show that starting with any point (0,0), we can find the length to the point (a,b) by drawing a right triangle; the length of the line segment from (0,0) to (a,b) will equal $\sqrt{a^2 + b^2}$.

D. Here's another formula that will come in handy later: To calculate the length of a line that connects any two points, (x_1,y_1) and (x_2,y_2), we can draw a right triangle that has a base of length $x_2 - x_1$ and a height of $y_2 - y_1$. According to the Pythagorean theorem, L^2, L being the length of the line from (x_1,y_1) to (x_2,y_2), is equal to $(x_2 - x_1)^2 + (y_2 - y_1)^2$. The length of the line would be the square root of that quantity.

E. Let's look at a problem that might seem a little challenging: We want to stretch a rope from the floor in one corner of the room to the ceiling in the opposite corner. What will the length of that rope have to be?

1. Let's put the question in more mathematical terms: We want to calculate the length of a line from the point (0,0,0) in three dimensions to the point (a,b,c).

2. Instead of looking at the point (a,b,c), let's look at the point $(a,b,0)$. We know from what we calculated earlier that the length of a line from point (0,0) to point (a,b) on the plane is $\sqrt{a^2 + b^2}$.
3. Picture a triangle whose base runs across the floor, whose length on one side is $\sqrt{a^2 + b^2}$, and whose length on the other side is c. All we're calculating now is the hypotenuse of that triangle. The length that we're interested in must satisfy $L^2 = a^2 + b^2 + c^2$.
4. Taking the square root of both sides, we find that the length of the line from (0,0,0) to (a,b,c) is $\sqrt{a^2 + b^2 + c^2}$.

F. Let's do one more problem: Imagine we start off with two squares, each of length 1, and we put them next to each other. Each has length 1; therefore, they are 1-by-1 squares with an area of 1.

1. Next, we'll build on that rectangle by adding other squares with dimensions that are Fibonacci numbers.
2. What is the area of the resulting rectangle? The right side has length 8, and the top length is 8 + 5, or 13; thus the area is 8×13.
3. We could also calculate the area of the rectangle by adding up the areas of all the individual squares. The sum of those areas is $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2$, or 8×13, the product of Fibonacci numbers.
4. We've now proved a pattern that we saw earlier; that is, the sum of the squares of the first n Fibonacci numbers is $F_n \times F_{n+1}$.

V. In our next lecture, we'll turn to the magical number pi.

Reading:

William Dunham, *Journey through Genius: The Great Theorems of Mathematics*, chapters 1–5.

Andrei Petrovich Kiselev, *Kiselev's Geometry, Book 1: Planimetry*, trans. by Alexander Givental.

Questions to Consider:

1. Show that the area of a parallelogram with horizontal base length b and vertical height h has area bh by rearranging a rectangle with the same dimensions.
2. Suppose that you tie a long rope to the bottom of the goalpost at one end of a football field. Then, you run it across the length of the field (120 yards) to a goalpost at the other end, stretch it tight, and tie it to the bottom of that goalpost so that it lies flat on the ground. Now suppose you add just 1 foot of slack to the rope so that you can lift it off the ground at the 50-yard line. How high can the rope be lifted up?

Lecture Twelve

The Joy of Pi

Scope: The previous lecture focused on geometrical objects that can be drawn with straight lines, but this lecture looks at the measurement of circles. We start with the observation that the ratio of the circumference and the diameter of any circle is always the same number, namely pi (π), a number that is slightly greater than 3. We use this to derive the area of the circle: πr^2, where r denotes the circle's radius. Pi is approximately equal to 22/7 or even closer to 3.1415926. We say "approximately" because it can be shown that pi is an irrational number, and thus, its decimal expansion will never repeat. Some people like to memorize digits of pi as a mental exercise, and we will learn a technique that allows you to learn its first 100 digits as easy as pie.

Outline

I. Let's begin this lecture on pi (π) by defining some terms.

A. The *radius* (r) of a circle is the distance from the center of the circle to the edge of the circle. The *diameter* of a circle is the distance obtained by drawing a line from one side of a circle to the other side through the center of the circle. The diameter is twice the radius ($d = 2r$). The *circumference* is the distance around the outside of the circle.

B. Surprisingly, if we divide the circumference of any circle by its diameter, we always get the same number, the constant ratio called pi, (written with the Greek letter π), or about 3.14.

C. Once we know the definition of *pi* (the ratio of the circumference to the diameter of any circle), we can calculate other quantities. For instance, the area of a circle is πr^2. We can prove this theorem in two ways.

1. Imagine that you have a circle in front of you and you cut through the top of the circle until you hit the radius. You then peel the circle away like an onion.
2. You unwrap the first layer of the circle and lay it down flat; that layer then has a length of $2\pi r$. Then, you peel off the next layer. That next layer has a length that's a little bit less than $2\pi r$. You continue to peel off layers until you can't peel the circle any longer.
3. Once you hit the center point at r, you have a triangle. We know that the area of a triangle is $1/2bh$. The base of the triangle has length $2\pi r$. The height of the triangle is r. Thus, the area of the triangle is $(1/2)(2\pi r)(r)$, which is πr^2.

D. Let's try another proof of this theorem. Imagine that you slice the circle up like a pizza into lots of triangles. Then, you separate the top half from the bottom half. The two sets of triangles can interlock to form a shape that's almost exactly a rectangle. The length of the bottom of the rectangle is πr because it came from half of the circumference. The length of the top of the rectangle is πr and the side length is r. Thus, the area of the rectangle is $\pi r(r)$, which is πr^2.

II. These proofs don't tell us why pi should be the number 3.14…. Here's one way to get a handle on the size of the number pi.

A. Let's look at a circle that has diameter 1. Remember that pi is the ratio of the circumference to the diameter; thus, if we have a circle with diameter 1, then pi will be the circumference of the circle.

B. Next, we draw a square inside the circle. Do you agree that the perimeter of that square is less than the perimeter of the circle? If we can figure out the perimeter of that square, then we'll have a lower bound for the perimeter of the circle.

1. Let's break that square, or diamond, into four right triangles and look at just one of those triangles. The triangle will have two sides whose lengths are $\frac{1}{2}$ because the radius of the circle was 1. According to the Pythagorean theorem, the length of the hypotenuse, when squared, will be $\frac{1}{2}^2 + \frac{1}{2}^2$, or $\frac{1}{4} + \frac{1}{4}$, or $\frac{2}{4}$.
2. We take $\frac{\sqrt{2}}{\sqrt{4}}$, or $\frac{\sqrt{2}}{2}$, to get the hypotenuse. Therefore, the perimeter of the diamond we drew in the middle of the circle is $4\left(\frac{\sqrt{2}}{2}\right)$, which is $2\left(\sqrt{2}\right)$, or about 2.828. We know, then, that the perimeter of the circle is larger than 2.828.
3. Finding an upper bound for the size of pi is even easier. We now put the circle inside of a square. The diameter of the circle is 1, which means that it will fit inside of a 1-by-1 square. The perimeter of a square with side length 1 is 4. Thus, the circumference of the circle must be less than 4.
4. We now have a lower bound and an upper bound for pi. We showed that pi, whatever it is, is somewhere between 2.828 and 4.

C. If we were to expand on this work, we could get better bounds for pi. In fact, the great mathematician Archimedes did so, using the same logic that we used, except instead of using 4-sided squares, he used 96-sided polygons to show that pi was between 3.1408 and 3.1428.

III. If we were to write pi out, it would be 3.141592653589…, going on forever. In 1761, Johann Heinrich Lambert proved that attempts to find pi exactly are futile. Pi is irrational, which means that it cannot be written as a fraction and its decimal expansion will never repeat.

IV. We know that pi is connected to a circle, but what about other shapes?

A. Let's look at an *ellipse*, which has the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. In other words, the points (x,y) that lie on the ellipse satisfy this equation.

1. In a drawing of an ellipse, the ellipse touches the x-axis when $x = a$ and when $x = -a$. The ellipse touches the y-axis when $y = b$ and $-b$. If we plug in $x = a$ and $y = 0$, we get $\frac{a^2}{a^2}+\frac{0^2}{b^2}$, which is 1.
2. The area of an ellipse is πab. If a and b are equal, then the ellipse becomes a circle. If a is r and b is r, then the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ becomes $\frac{x^2}{r^2}+\frac{y^2}{r^2}=1$. When we multiply that by r^2, the equation becomes $x^2 + y^2 = r^2$, which is the formula for the area of a circle of radius r. In that case, the area of the circle would be $\pi(r)(r)$, or πr^2.

B. We also find pi in the volume of a cylinder that has a circular base of radius r and a height of h. Think of a can of soup. The base of the can has an area of πr^2 and it is then raised up to a height of h; obviously, we have to multiply h by πr^2 to get a volume of $\pi r^2 h$.

1. To calculate the surface area, we have to calculate the area of the top and bottom of the can and the area that goes around the can. The areas of the top and bottom of the can are πr^2.
2. If we were to unwrap the can and flatten it out, it would still have a height of h and its length would be the original circumference of the can, which is $2\pi r$. Thus, the area of the rectangle we get when we flatten out the can is $2\pi rh$. When we put it all together, the surface area is $2\pi r^2 + 2\pi rh$.

C. The volume of a right circular cone is $\frac{\pi r^2 h}{3}$. Think of an upside-down ice-cream cone. We have a circle of radius r on the bottom, and the cone goes straight up to a height of h, then down again to the circle. Exactly three of those cones could fit into a cylinder. The surface area of a right circular cone is $\pi r^2 + \sqrt{r^2+h^2}$.

D. The volume of a sphere of radius r is $\frac{4\pi r^3}{3}$, and the surface area of a sphere is $4\pi r^2$. Those are best derived using calculus.

V. Pi also appears in more unusual places.

A. For example, the sum $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$, gets closer to $\frac{\pi^2}{6}$ exactly. The sum $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots$, gets closer to $\frac{\pi^4}{90}$.

B. Pi also intersects number theory. If we pick two enormous numbers, a and b, at random, the probability that the greatest common divisor of a and b is 1 is exactly $\frac{6}{\pi^2}$, about 60 %.

C. Another number that we saw earlier in our lectures was $n!$, the number of ways that we can arrange n objects. Believe it or not, $n!$ has an approximation that uses pi. Especially when n is large, this approximation is almost exactly equal to $\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$. The number e is about 2.71828. We'll see more about that number later.

D. Like phi, the golden ratio, pi has a continued fraction that goes on forever, shown at right.

$$\pi = 3+\cfrac{1}{6+\cfrac{9}{6+\cfrac{25}{6+\cfrac{49}{6+\cfrac{81}{\cdots}}}}}$$

E. Pi even has a connection to the Fibonacci numbers, especially when we study trigonometry. Look at the formula below. As we add up more of those arc tangents and skip every other Fibonacci number, we get closer to $\pi/4$.

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\tan^{-1}\left(\frac{1}{34}\right)\ldots$$

F. When we talk about probability later on, we'll encounter the famous bell curve, which has a height of $\frac{1}{\sqrt{2\pi}}$.

VI. Pi is often celebrated in fun ways.

A. For instance, many people enjoy events on "Pi Day," which is celebrated on March 14 at 1:59, or 314159.

B. The world record right now for memorizing pi is more than 40,000 digits. One way to memorize pi involves a paraphrase of Edgar Allen

Poe's poem "The Raven" written by Mike Keith. In Keith's poem, the number of letters in each word equates to the digits of pi.

C. You can also memorize the first 24 digits of pi using this sentence: "My turtle Pancho will, my love, pick up my new mover, Ginger."

1. This sentence uses a phonetic code, in which every digit has an associated consonant sound, as shown in the following table:

1	t or d	6	j, ch, or sh
2	n	7	k or hard g
3	m	8	f or v
4	r	9	p or b
5	L	0	s or z

Note: the consonants for h, w, and y are not represented in this code.

(A possible mnemonic is "Danny Marloshkovips.")

2. Look at the sentence about Pancho and the first five digits of pi (31415). By inserting vowel sounds, we turn 3 into the word *my*; then, for 1415, the t, r, t, and l sounds become *turtle*. Continuing this process, the sentence translates to the first 24 digits of pi.
3. The next 17 digits correspond to "My movie monkey plays in a favorite bucket," and the next 19 digits match with "Ship my puppy Michael to Sullivan's back-rubber." If we want to go up to 100 digits, then the next 40 digits correspond to these two sentences: "A really open music video cheers Jenny F. Jones," followed by, "Have a baby fish knife so Marvin will marinate the goosechick."

Reading:

David Blatner, *The Joy of Pi*.

Y. E. O. Adrian, *The Pleasures of Pi, e and Other Interesting Numbers*.

Joy of Pi, www.joyofpi.com/.

Arthur Benjamin and Michael Shermer. *Secrets of Mental Math*, chapter 7.

Questions to Consider:

1. Suppose you have a rope around the equator of a basketball. How much longer would you have to make the rope so that it is 1 foot from the surface of the basketball at all points? The answer is 2π feet. Now suppose you have the rope around the equator of the Earth. (Yes, a rope about 25,000 miles long!) How much longer would you have to make that rope so that it is 1 foot off the ground all the way around the equator?
2. Starting with the famous formula for the sum of squares of reciprocals: $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots=\frac{\pi^2}{6}$, derive a formula for the sum of the squares of the even reciprocals and the sum of the squares of the odd reciprocals.

Glossary

algebra: Literally, the reunion of broken parts; the manipulation of both sides of an equation to solve for an unknown quantity.

algebraic proof: Establishing the truth of a statement through algebraic manipulation.

anti-derivative: A function whose derivative is a given function.

axiom: A statement that is accepted without proof, such as: "For any two points, there is exactly one line that goes through them."

binomial probability: If an experiment is performed n times, and each experiment independently has a probability p of success, then this is the probability that exactly k successes will occur; numerically equal to $\binom{n}{k}p^k(1-p)^{n-k}$.

binomial theorem: How to expand $(x+y)^n$; the coefficients of the expansion appear on Pascal's triangle. More precisely, it says: $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.

calculus: The branch of mathematics that deals with limits and the differentiation and integration of functions of one or more variables. See also **differential calculus** and **integral calculus**.

central limit theorem of probability: The average of a large number of random variables tends to have a normal (bell-shaped) distribution.

circumference: The perimeter of a circle.

combinatorics: The mathematics of enumeration; the only subject that really counts.

combinatorial proof: Establishing the truth of a statement by counting a set in two different ways.

complex number: A number of the form $a + bi$, where i is an imaginary number.

composite number: A positive number with three or more divisors.

conditional probability: The probability that an event occurs, given that another event has occurred.

cosine: For a given angle a, cosine a, is the x-coordinate of the point on the unit circle associated with angle a.

derivative: The rate of change of a function at a given point.

diameter: The length of a line segment obtained by drawing a line from one side of a circle through the center of the circle to the other side of the circle.

differential calculus: The mathematics of how things change and grow.

differential equation: An equation satisfied by a function and its derivatives. For example, the function $y = e^{kx}$ satisfies the differential equation $y' = ky$.

differentiation: The process of calculating derivatives.

e: A number of "exponential" importance, the number e is equal to 2.71828…, which is the limit of $(1 + 1/n)^n$ as n approaches infinity.

equilateral triangle: A triangle that has three equal side lengths.

Euler's equation: A formula that brings algebra, geometry, and trigonometry together: $e^{ix} = \cos x + i \sin x$. When $x = \pi$, it follows that $e^{i\pi} + 1 = 0$.

exponent: The exponent of a^n is the number n. When n is positive, a^n equals a multiplied n times; when n is negative, a^n equals $1/a$ multiplied n times; $a^0 = 1$.

factorial: The number $n!$ is the product of the numbers from 1 through n.

Fibonacci numbers: The numbers obtained in the sequence 1, 1, 2, 3, 5, 8, 13,…, where each number is the sum of the previous two numbers.

fundamental theorem of algebra: States that the graph of a polynomial of degree n will intersect the x-axis at most n times; more precisely, a polynomial of degree n has at most n complex roots.

fundamental theorem of arithmetic: Every positive number can be factored into prime numbers in a unique way.

fundamental theorem of calculus: For any positive function $y = f(x)$, the area under the curve $y = f(x)$ that lies above the x-axis and between a and b is equal to $F(b) - F(a)$ where $F(x)$ is a function with derivative $f(x)$.

geometric probability: If an experiment is performed until a success occurs, and each experiment has probability p of success, then this is the probability that the first success will occur on the n^{th} trial; numerically equal to $(1-p)^{n-1} p$.

geometric series: A useful infinite series that says for all numbers x with absolute value less than 1, $1 + x + x^2 + x^3 + \ldots = 1/(1 - x)$.

geometry: The mathematics of measurement.

golden ratio (phi): the value $\dfrac{1+\sqrt{5}}{2} = 1.618\ldots$, a number with many beautiful properties; in the limit, the ratio of ever larger consecutive Fibonacci numbers.

harmonic series: The infinite sum 1 + 1/2 + 1/3 + 1/4 + 1/5 + …, which diverges to infinity.

hyperbolic functions: $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$ are called hyperbolic functions because they satisfy $\cosh^2 x - \sinh^2 x = 1$, and therefore, $(\cosh x, \sinh x)$ is a point on the unit hyperbola. Also, $\tanh x = \sinh x/\cosh x$. Many relationships satisfied by hyperbolic functions are analogous to ones satisfied by the usual trigonometric functions of sine, cosine, and tangent.

i: The square root of negative one, located 1 unit above zero on the imaginary axis. It is one of two solutions to the equation $x^2 + 1 = 0$, the other solution being negative *i*.

imaginary number: The square root of a negative number.

induction, proof by: To prove that a statement is true for all positive integers, prove it for the number 1, and show that if it is true for the number k, then it will continue to be true for $k + 1$.

infinite series: The sum of infinitely many numbers. We say that an infinite sum of numbers converges to S means that as you add more and more terms you get closer to S, eventually getting as close as you want.

infinity: The number of numbers, larger than any number. (The more you contemplate it, the more your mind gets number!)

integer: A whole number, which can be positive, negative, or zero.

integral calculus: The mathematics of determining a quantity, such as volume or area, by breaking the quantity into very small parts.

integration: The process used to calculate areas and volumes by making use of the fundamental theorem of calculus.

isosceles triangle: A triangle with two sides of equal length.

law of cosines: For any triangle with side lengths a, b, c: $c^2 = a^2 + b^2 - 2ab \cos C$, where C is the angle opposite side c.

law of sines: For any triangle with side lengths a, b, c with corresponding angles A, B, C, $(\sin A)/a = (\sin B)/b = (\sin C)/c$.

law of total probability: The probability that an event A occurs can be determined by first considering whether or not another event B occurs: specifically, P(A) = P(A|B)P(B) + P(A|not B) P(not B), where P(A|B) denotes the probability that A occurs, given that B occurs.

logarithm: The exponent needed to obtain one number from another. More precisely, the base b logarithm of a is the number x that satisfies $b^x = a$. The power of 10 needed to obtain a given number is called the base-10 logarithm; for example, the base 10 logarithm of 1,000 is 3.

modular arithmetic: The mathematics of remainders.

normal distribution: Popularly known as the bell-shaped curve, a random variable with a normal distribution has about a 68% chance of being within one standard deviation away from its mean and about a 95% chance of being within two standard deviations away from its mean.

perfect number: A number that is equal to the sum of all its proper divisors. For example, 6 is perfect because $6 = 1 + 2 + 3$.

pi: The ratio of the circumference of any circle to its diameter, denoted by the Greek letter π.

polynomial: A sum of terms of the form ax^n where the number a is called the coefficient and the exponent n must be an integer greater than or equal to zero.

prime number: A positive number that has exactly two divisors, 1 and itself.

probability: The likelihood of an event. An event with probability near 1 is nearly certain; an event with probability near 0 is nearly impossible.

Pythagorean theorem: In any right triangle with side lengths a, b, c:

$a^2 + b^2 = c^2$, where c is the length of the hypotenuse.

quadratic formula: The equation $ax^2 + bx + c = 0$ has the solution

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The word "quadratic" comes from the word for "square."

radian: The angle equal to $180/\pi$ degrees.

radius: The distance from the center of a circle to the edge of the circle; equal to 1/2 the diameter of the circle.

rational number: A number that can be expressed as the ratio of two integers.

reciprocal function: A function times its reciprocal function is 1. For example, the reciprocal of cos x is 1/cos x (also known as sec x).

second-degree equation: A function of the form $y = ax^2 + bx + c$.

sine: For a given angle a, sine a, is the y-coordinate of the point on the unit circle associated with angle a.

tangent: Sine divided by cosine.

theorem: A mathematical truth derivable from axioms and the rules of logic.

trigonometry: The branch of mathematics that deals with the relationships between the sides and angles of triangles.

variable: A non-constant numerical quantity.

variance: Measures how much the values of a variable spread around the mean of that variable. Square root of the variance is known as the standard deviation.

Notes